Ill-Posed Initial-Boundary Value Problems for Linear Hyperbolic Systems of Second Order

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In the rectangle $\Omega=[0,\omega_1]\times[0,\omega_2]$ consider the initial-boundary value problem

$$u_{xy}=P_0(x,y)u+P_1(x,y)u_x+P_2(x,y)u_y+q(x,y), \tag{1}$$

$$u(0,y)=\varphi(y), \quad h(u_x(x,\,\cdot\,))(x)=\psi(x), \tag{2}$$

where $P_j\in C(\Omega;\mathbb{R}^{n\times n})$ $(j=0,1,2)$, $q\in C(\Omega;\mathbb{R}^n)$, $\varphi\in C^1([0,\omega_2];\mathbb{R}^n)$, $\psi\in C([0,\omega_1];\mathbb{R}^n)$, and $h:C([0,\omega_2];\mathbb{R}^n)\to C([0,\omega_1];\mathbb{R}^n)$ is a bounded linear operator.

We make use of the following notations:

- I_m is $m\times m$ identity matrix, O_m is the $m\times m$ zero matrix, $O_{m,k}$ is $m\times k$ zero matrix.
- If $A=(a_{ij}))_{i,j=1}^n\in\mathbb{R}^{n\times n}$, then by $[A]_{m,m}$ we denote its principal $m\times m$ submatrix, i.e.

$$[A]_{m,m}=(a_{ij})_{i,j=1}^m.$$

- If $z=(z_i)_{i=1}^n\in\mathbb{R}^n$, then $[z]^m=(z_i)_{i=1}^m$ and $[z]_m=(z_i)_{i=m+1}^n$.
- $C^{m,k}(\Omega;\mathbb{R}^n)$ is the Banach space of functions $u:\Omega\to\mathbb{R}^n$, having continuous partial derivatives $u^{(i,j))}$ $(i=0,\dots,m;\ j=0,\dots k)$, endowed with the norm

$$\|u\|_{C^{\mathbf{m}}(\Omega)}=\sum_{i=0}^m\sum_{j=0}^k\Big\|\frac{\partial^{i+j}u}{\partial x^i\partial y^j}\Big\|_{C(\Omega)}.$$

By a solution of problem (1), (2) we understand a *classical* solution, i.e., a function $u\in C^{1,1}(\Omega)$ satisfying equation (1) and the boundary conditions (2) everywhere in Ω.

Along with problem (1), (2) consider the problem

$$v'=P_1(x^*,y)v, \tag{3}$$

$$h(v)(x^*)=0. \tag{4}$$

Problem (3), (4) is called ***associated problem*** of problem (1), (2). Notice that problem (3), (4) is a boundary value problem for a linear ordinary differential equation depending on the parameter x^*.

The associate problem (3), (4) plays a decisive in the study of problem (1), (2). Theorems 4.1 and 4.1′ from [1] state that if for every $x^*\in[0,\omega_1]$ problem (3), (4) has only the trivial solution, then problem (1), (2) is well-posed, i.e., it is uniquely solvable for arbitrary $\varphi\in C^1([0,\omega_2];\mathbb{R}^n)$, $\psi\in C([0,\omega_1];\mathbb{R}^n)$ and $q\in C(\Omega;\mathbb{R}^n)$, and its solution u admits the estimate

$$\|u\|_{C^{1,1}(\Omega)}\leq M\Big(\|\varphi\|_{C^1([0,\omega_2])}+\|\psi\|_{C([0,\omega_1])}+\|q\|_{C(\Omega)}\Big), \tag{5}$$

where M is a positive constant independent of φ, ψ and q.

In [3] the inverse statement is proved: if problem (1), (2) is well-posed, i.e., if a solution of problem (1), (2) admits estimate (5), then the associate problem (3), (4) has only the trivial solution for every $x^* \in [0, \omega_1]$.

Well-posed initial-boundary value problems and nonlocal boundary value problems for linear hyperbolic systems were studied in [1] and [3]. Well-posed initial-boundary value problems and nonlocal boundary value problems for higher order linear hyperbolic equations were studied in [4] and [2]. Ill-posed initial-boundary value problems for higher order linear hyperbolic equations were studied in [5].

Let $Y(x^*, y)$ be the fundamental matrix of differential system (3) such that $Y(x^*, 0) = I$.

If $h : C([0, \omega_2]; \mathbb{R}^n) \to C([0, \omega_1]; \mathbb{R}^n)$ is a bounded linear operator, then according to Lemmas 2.1_1 and 2.3_1 from [1],

$$h(z)(x) = H(x)z(0) + \int_0^{\omega_2} K(x,t)v'(t)\,dt,$$

$$h(Y(x,\,\cdot\,)z(\,\cdot\,))(x) = M_0(x)z(0) + \int_0^{\omega_2} M(x,t)v'(t)\,dt,$$

where

$$H(x) \in C([0, \omega_1]; \mathbb{R}^{n\times n}), \quad K \in L^\infty(\Omega; \mathbb{R}^{n\times n}),$$

$$M_0(x) = H(x) + \int_0^{\omega_2} M(x,t)Y_t(x,t)\,dt,$$

$$M(x,y) = K(x,y)Y(x,y) + \int_y^{\omega_2} K(x,t)Y_t(x,t)\,dt.$$

If $h : C([0, \omega_2]; \mathbb{R}^n) \to C^1([0, \omega_1]; \mathbb{R}^n)$ is a bounded linear operator, then

$$H(x) \in C^1([0, \omega_1]; \mathbb{R}^{n\times n}),$$

$$M_0(x) \in C^1([0, \omega_1]; \mathbb{R}^{n\times n}), \qquad \int_0^{y} M(x,t)\,dt \in C^{1,1}(\Omega; \mathbb{R}^{n\times n}).$$

In terms of the matrix H, problem (1), (2) is well-posed if and only if $\operatorname{rank} H(x) = n$ for every $x \in [0, \omega_1]$. The ill-posed problem (1), (2) in the case where $\operatorname{rank} H(x) \equiv 0$ was studied in [1].

In this paper, we study the ill-posed problem (1), (2) in case where $\operatorname{rank} H(x) = n - m$ for some $m \in \{1, \dots, n\}$. For the sake of technical simplicity we will assume that

$$H(x) = \begin{pmatrix} O_m & O_{m,n-m} \\ O_{n-m,m} & H_0(x) \end{pmatrix}, \quad \operatorname{rank} H_0(x) = n - m \ \text{ for } \ x \in [0, \omega_1]. \tag{6}$$

Theorem 1. *Let* (6) *hold, let* $P_1 \in C^{1,0}(\Omega; \mathbb{R}^n)$ *and let*

$$\det\Big[\int_0^{\omega_2} M(x,t)Z^{-1}(x,t)\big(P_0(x,t) + P_2(x,t)P_1(x,t)\big)Z(x,t)\,dt\Big]_{mm} \neq 0 \ \textit{ for } \ x \in [0, \omega_1]. \tag{7}$$

Then problem (1), (2) *is solvable in the weak sense if and only if the equality*

$$\Big[\int_0^{\omega_2} M(0,t)Z^{-1}(0,t)\big(P_0(0,t)\varphi(t)+P_2(0,t)\varphi'(t)+q(0,t)\big)\,dt\Big]_m=[\psi(0)]_m \tag{8}$$

holds. Moreover, if equality (8) *holds, then problem* (1), (2) *has a unique weak solution* $u\in C^{0,1}(\Omega;\mathbb{R}^n)$ *admitting the estimate*

$$\big\|[u]^m\big\|_{C^{0,1}(\Omega)}\le M\Big(\|q\|_{C(\Omega)}+\|\varphi\|_{C^1([0,\omega_2])}+\|\psi\|_{C([0,\omega_1])}\Big), \tag{9}$$

$$\big\|[u]_m\big\|_{C^{1,1}(\Omega)}\le M\Big(\|q\|_{C(\Omega)}+\|\varphi\|_{C^1([0,\omega_2])}+\|\psi\|_{C([0,\omega_1])}\Big), \tag{10}$$

where M *is a positive constant independent of* φ, ψ *and* q.

Theorem 2. *Let* $P_j\in C^{1,0}(\Omega;\mathbb{R}^{n\times n})$ $(j=0,1,2)$, $q\in C^{1,0}(\Omega;\mathbb{R}^n)$, $\psi\in C^1([0,\omega_1];\mathbb{R}^n)$, *let* $h:C([0,\omega_2];\mathbb{R}^n)\to C^1([0,\omega_1];\mathbb{R}^n)$ *be a bounded linear operator, and let conditions* (6) *and* (7) *hold. Then problem* (1), (2) *is solvable in the classical sense if and only if equality* (8) *holds. Moreover, if equality* (8) *holds, then problem* (1), (2) *has a unique classical solution* u *admitting the estimate*

$$\big\|[u]^m\big\|_{C^{1,1}(\Omega)}\le M\Big(\|q\|_{C^{1,0}(\Omega)}+\|\varphi\|_{C^1([0,\omega_2])}+\|\psi\|_{C^1([0,\omega_1])}\Big), \tag{11}$$

$$\big\|[u]_m\big\|_{C^{2,1}(\Omega)}\le M\Big(\|q\|_{C^{1,0}(\Omega)}+\|\varphi\|_{C^1([0,\omega_2])}+\|\psi\|_{C^1([0,\omega_1])}\Big), \tag{12}$$

where M *is a positive constant independent of* φ, ψ *and* q.

Remark 1. Estimates (9), (10), (11) and (12) are sharp and cannot be relaxed. Indeed, let $n=2m$, $u=(v,w)$, $v,w\in\mathbb{R}^m$. Consider the problem

$$\begin{aligned} v_{xy}&=P(x)v+q_1(x),\\ w_{xy}&=Q_1(x,y)v+Q_2(x,y)w+q_2(x,y),\end{aligned} \tag{13}$$

$$v(0,y)=c,\quad w(0,y)=\varphi_2(y),\quad v_x(x,0)=v_x(x,\omega_2),\quad w_x(x,0)=0. \tag{14}$$

For problem (13), (14)

$$H(x)=\begin{pmatrix} O_m & O_m\\ O_m & I_m\end{pmatrix}.$$

Assume that problem (13), (14) has a unique weak solution $(v(x,y),w(x,y))\in C^{0,1}(\Omega;\mathbb{R}^{2m})$. Then $w(x,y)$ is a solution of the integral equation

$$w(x,y)=\varphi_2(y)+\int_0^x\int_0^y\big(Q_2(s,t)w(s,t)+Q_1(s,t)v(s,t)+q_2(s,t)\big)\,dt\,ds. \tag{15}$$

The integral equation (15) is uniquely solvable for every $v(x,y)\in C(\Omega;\mathbb{R}^m)$, and its solution belongs to $C^{1,1}(\Omega;\mathbb{R}^m)$.

Consequently, the unique solvability of problem (13), (14) is equivalent to the unique solvability of the problem

$$v_{xy}=P(x)v+q_1(x), \tag{16}$$

$$v(0,y)=c,\quad v_x(x,0)=v_x(x,\omega_2). \tag{17}$$

Let $v \in C^{0,1}(\Omega;\mathbb{R}^m)$ be a weak solution of problem (16), (17). It admits a continuous ω_2-periodic continuation with respect to y. It is clear that $v(x,y+r)$ is a solution of problem (16), (17) for every $r \in [0,\omega_1]$. Therefore, the unique solvability of problem (16), (17) implies that

$$v(x,y) \equiv v(x).$$

Consequently, v is a solution of the linear algebraic system

$$P(x)v(x) + q_1(x) = 0.$$

The latter system is uniquely solvable for an arbitrary $q_1(x)$ if and only if

$$\det P(x) \neq 0 \ \text{ for } \ x \in [0,\omega_1],$$

i.e., inequality (7) holds for problem (13), (14). Thus problem (16), (17) has the unique weak solution

$$v(x) = -P^{-1}(x)q_1(x)$$

if and only if

$$c = -P^{-1}(0)q(0).$$

This solution is classical if $P \in C^1([0,\omega_1];\mathbb{R}^{m\times m})$ and $q_1 \in C^1([0,\omega_1];\mathbb{R}^m)$. This confirms the sharpness of estimates (9), (10), (11) and (12).

If $\det P(x^*) = 0$ for some $x^* \in [0,\omega_1]$, then problem (16), (17) may not have a weak solution even if $P \in C^\infty(\omega;\mathbb{R}^{m\times m})$ and $q \in C^\infty(\omega;\mathbb{R}^m)$.

Finally, notice that if $\det P(0) \neq 0$, $P(x) = O_m$ for $x \in [a,b]$ for some interval $[a,b] \subset (0,\omega_1)$, and $q(x) = P(x)\overline{q}(x)$, then problem (16), (17) and, consequently, problem (13), (14), have infinite-dimensional sets of solutions.

For the system

$$u_{xy} = P_0(x,y)u + P_2(x,y)u_y + q(x,y), \tag{18}$$

consider the initial-boundary conditions

$$u(0,y) = \varphi(y), \quad \big[u_x(x,\gamma_2(x)) - u_x(x,\gamma_1(x))\big]^m = [\psi(x)]^m, \quad [u_x(x,0)]_m = [\psi(x)]_m, \tag{19}$$

and

$$u(0,y) = \varphi(y), \quad u_x(x,0) = u_x(x,\omega_2). \tag{20}$$

Here $\gamma_i \in C([0,\omega_1])$ $(i=1,2)$ and $\gamma_1(x) < \gamma_2(x)$ for $x \in [0,\omega_1]$.

Theorem 3. *Let*

$$\det\Bigg[\int\limits_{\gamma_1(x)}^{\gamma_2(x)} P_0(x,t)\,dt\Bigg]_{mm} \neq 0 \ \textit{ for } \ x \in [0,\omega_1].$$

Then problem (18), (19) *is solvable in the weak sense if and only if the equality*

$$\Bigg[\int\limits_{\gamma_1(0)}^{\gamma_2(0)} \big(P_0(0,t)\varphi(t) + P_2(0,t)\varphi'(t) + q(0,t)\big)\,dt\Bigg]_m = [\psi(0)]_m \tag{21}$$

holds. Moreover, if equality (21) *holds, then problem* (18), (19) *has a unique weak solution* u *admitting the estimate*

$$\|u\|_{C^{0,1}(\Omega)} \leq M\Big(\|q\|_{C(\Omega)} + \|\varphi\|_{C^1([0,\omega_2])} + \|\psi\|_{C([0,\omega_1])}\Big),$$

where M is a positive constant independent of φ, ψ and q. Moreover, if $P_j \in C^{1,0}(\Omega;\mathbb{R}^{n\times n})$ $(j=0,2)$, $q \in C^{1,0}(\Omega;\mathbb{R}^n)$, $\psi \in C^1([0,\omega_1];\mathbb{R}^n)$ and $\gamma_i \in C^1([0,\omega_1])$ $(i=1,2)$, then u is a classical solution admitting the estimate

$$\|u\|_{C^{1,1}(\Omega)} \le M\Big(\|q\|_{C^{1,0}(\Omega)} + \|\varphi\|_{C^1([0,\omega_2])} + \|\psi\|_{C^{1,0}([0,\omega_1])}\Big).$$

Corollary 1. *Let*

$$\det \int\limits_0^{\omega_2} P_0(x,t)\,dt \neq 0 \quad \text{for } x \in [0,\omega_1].$$

Then problem (18), (20) is solvable if and only if the equality

$$\int\limits_0^{\omega_2} \big(P_0(0,t)\varphi(t) + P_2(0,t)\varphi'(t) + q(0,t)\big)\,dt = 0 \tag{22}$$

holds. Moreover, if equality (22) holds, then problem (18), (20) has a unique weak solution u admitting the estimate

$$\|u\|_{C^{0,1}(\Omega)} \le M\Big(\|q\|_{C(\Omega)} + \|\varphi\|_{C^1([0,\omega_2])}\Big),$$

where M is a positive constant independent of φ, ψ and q. Moreover, if $P_0 \in C^{1,0}(\Omega;\mathbb{R}^{n\times n})$ and $q \in C^{1,0}(\Omega;\mathbb{R}^n)$, then u is a classical solution admitting the estimate

$$\|u\|_{C^{1,1}(\Omega)} \le M\Big(\|q\|_{C^{1,0}(\Omega)} + \|\varphi\|_{C^1([0,\omega_2])}\Big).$$

Let $n = 2m$, $u = (v,w)$, and $v, w \in \mathbb{R}^m$. For the systems

$$\begin{aligned} v_{xy} &= w_x + B_{11}(x,y)v_y + B_{12}(x,y)w_y + Q_{11}(x,y)v + Q_{12}(x,y)w + q_1(x,y),\\ w_{xy} &= -v_x + B_{21}(x,y)v_y + B_{22}(x,y)w_y + Q_{21}(x,y)v + Q_{22}(x,y)w + q_2(x,y) \end{aligned} \tag{23}$$

and

$$\begin{aligned} v_{xy} &= w_x + B(x,y)v_y + Q(x,y)v + q_1(x,y),\\ w_{xy} &= -v_x + B(x,y)w_y + Q(x,y)w + q_2(x,y), \end{aligned} \tag{24}$$

consider the initial-periodic conditions

$$v(0,y) = \varphi_1(y), \quad w(0,y) = \phi_2(y), \quad v_x(x,0) = v_x(x,2\pi), \quad w_x(x,0) = w_x(x,2\pi). \tag{25}$$

Theorem 4. *Let*

$$\det\left(\int\limits_0^{2\pi} \begin{pmatrix} \cos t\, I_m & \sin t\, I_m \\ -\sin t\, I_m & \cos t\, I_m \end{pmatrix} \begin{pmatrix} Q_{11}(x,t) - B_{12}(x,t) & Q_{12}(x,t) + B_{11}(x,t) \\ Q_{21}(x,t) - B_{12}(x,t) & Q_{11}(x,t) - B_{12}(x,t) \end{pmatrix} \times \begin{pmatrix} \cos t\, I_m & -\sin t\, I_m \\ \sin t\, I_m & \cos t\, I_m \end{pmatrix} dt\right) \neq 0 \quad \text{for } x \in [0,\omega_1].$$

Then problem (23), (25) is solvable in the weak sense if and only if the equality

$$\int\limits_0^{2\pi} \begin{pmatrix} \cos t\, I_m & -\sin t\, I_m \\ \sin t\, I_m & \cos t\, I_m \end{pmatrix} \times \begin{pmatrix} Q_{11}(0,t)\varphi_1(t) + Q_{12}(0,t)\varphi_2(t) + B_{11}(0,t)\varphi_1'(t) + B_{12}(0,t)\varphi_2'(t) + q_1(0,t) \\ Q_{21}(0,t)\varphi_1(t) + Q_{22}(0,t)\varphi_2(t) + B_{21}(0,t)\varphi_1'(t) + B_{22}(0,t)\varphi_2'(t) + q_2(0,t) \end{pmatrix} dt = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{26}$$

holds. Moreover, if equality (26) *holds, then problem* (23), (25) *has a unique weak solution* (v,w) *admitting the estimate*

$$\|v\|_{C^{0,1}(\Omega)} + \|w\|_{C^{0,1}(\Omega)} \le M\Big(\|q\|_{C(\Omega)} + \|\varphi\|_{C^1([0,\omega_2])}\Big),$$

where M *is a positive constant independent of* φ, ψ *and* q. *Moreover, if* $B_{ij} \in C^{1,0}(\Omega;\mathbb{R}^{m\times m})$ *and* $Q_{ij} \in C^{1,0}(\Omega;\mathbb{R}^{m\times m})$ $(i,j=1,2)$, *and* $q_i \in C^{1,0}(\Omega;\mathbb{R}^m)$ $(i=1,2)$, *then* (v,w) *is a classical solution admitting the estimate*

$$\|v\|_{C^{1,1}(\Omega)} + \|w\|_{C^{1,1}(\Omega)} \le M\Big(\|q\|_{C^{1,0}(\Omega)} + \|\varphi\|_{C^1([0,\omega_2])}\Big).$$

Corollary 2. *Let*

$$\det\left(\int\limits_0^{2\pi} \begin{pmatrix} Q(x,t) & B(x,t) \\ -B(x,t) & Q(x,t) \end{pmatrix} dt\right) \neq 0 \;\; \text{for} \;\; x \in [0,\omega_1].$$

Then problem (23), (24) *is solvable in the weak sense if and only if the equality*

$$\int\limits_0^{2\pi} \begin{pmatrix} \cos t\, I_m & -\sin t\, I_m \\ \sin t\, I_m & \cos t\, I_m \end{pmatrix} \begin{pmatrix} Q(0,t)\varphi_1(t) + B(0,t)\varphi_1'(t) + q_1(0,t) \\ Q(0,t)\varphi_2(t) + B(0,t)\varphi_2'(t) + q_2(0,t) \end{pmatrix} dt = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

holds. Moreover, if equality (26) *holds, then problem* (23), (24) *has a unique weak solution* u *admitting the estimate*

$$\|v\|_{C^{0,1}(\Omega)} + \|w\|_{C^{0,1}(\Omega)} \le M\Big(\|q\|_{C(\Omega)} + \|\varphi\|_{C^1([0,\omega_2])}\Big),$$

where M *is a positive constant independent of* φ, ψ *and* q. *Moreover, if* $B \in C^{1,0}(\Omega;\mathbb{R}^{m\times m})$, $Q \in C^{1,0}(\Omega;\mathbb{R}^{m\times m})$, *and* $q_i \in C^{1,0}(\Omega;\mathbb{R}^m)$ $(i=1,2)$, *then* u *is a classical solution admitting the estimate*

$$\|v\|_{C^{1,1}(\Omega)} + \|w\|_{C^{1,1}(\Omega)} \le M\Big(\|q\|_{C^{1,0}(\Omega)} + \|\varphi\|_{C^1([0,\omega_2])}\Big).$$

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