Antiperiodic in Time Boundary Value Problem for One Class of Nonlinear High-Order Partial Differential Equations

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In the plane of variables x and t consider a nonlinear high−order partial differential equation of the form

$$L_f u := \frac{\partial^2 u}{\partial t^2} - \frac{\partial^{4k} u}{\partial x^{4k}} + f(u) = F, \tag{1}$$

where f, F are given, while u is an unknown functions, k is a natural number.

For the equation (1) we consider the following antiperiodic in time problem: find in the domain $D_T: \ 0 < x < l, \ 0 < t < T$ a solution $u = u(x,t)$ of the equation (1) according to the boundary conditions

$$u(x,0) = -u(x,T), \quad u_t(x,0) = -u_t(x,T), \ \ 0 \leq x \leq l, \tag{2}$$

$$\frac{\partial^i u}{\partial x^i}\,(0,t) = 0, \quad \frac{\partial^i u}{\partial x^i}\,(l,t) = 0, \ \ 0 \leq t \leq T, \ \ i = 0,\dots,2k-1. \tag{3}$$

Note that to the study of antiperiodic and periodic problems for nonlinear partial differential equations, having a structure different from (1), is devoted numerous literature (see, for example, [1, 2, 4–8] and the literature cited therein). For the equation (1) with $k = 1$, antiperiodic problem, both in terms of time and space variables, is considered in the work [3].

Denote by $C^{2,4k}(\overline{D}_T)$ the space of functions continuous in $\overline{D}_T$, having in $\overline{D}_T$ continuous partial derivatives $\frac{\partial^i u}{\partial t^i}$, $i = 1,2$, $\frac{\partial^j u}{\partial x^j}$, $j = 1,\dots,4k$. Let

$$C_0^{2,4k}(\overline{D}_T) := \Big\{ u \in C^{2,4k}(\overline{D}_T): \ \ \frac{\partial^i u}{\partial t^i}\,(x,0) = -\frac{\partial^i u}{\partial t^i}\,(x,T), \ \ 0 \leq x \leq l, \ \ i = 0,1;$$
$$\frac{\partial^j u}{\partial x^j}\,(0,t) = 0, \ \ \frac{\partial^j u}{\partial x^j}(l,t) = 0, \ \ 0 \leq t \leq T, \ \ j = 0,\dots,2k-1 \Big\}.$$

Consider the Hilbert space $W_0^{1,2k}(D_T)$ as a completion of the classical space $C_0^{2,4k}(\overline{D}_T)$ with respect to the norm

$$\|u\|^2_{W_0^{1,2k}(D_T)} = \int\limits_{D_T} \bigg[u^2 + \Big(\frac{\partial u}{\partial t}\Big)^2 + \sum_{i=1}^{2k} \Big(\frac{\partial^i u}{\partial x^i}\Big)^2 \bigg]\,dx\,dt. \tag{4}$$

It follows from (4) that if $u \in W_0^{1,2k}(D_T)$, then $u \in W_2^1(D_T)$ and $\frac{\partial^i u}{\partial x^i} \in L_2(D_T)$, $i = 2,\dots,2k$. Here $W_2^1(D_T)$ is the well-known Sobolev space consisting of the elements $L_2(D_T)$, having up to the first order generalized derivatives from $L_2(D_T)$.

Remark 1. Let $u \in C_0^{2,4k}(\overline{D}_T)$ be a classical solution of the problem (1)–(3). Multiplying the both sides of the equation (1) by an arbitrary function $\varphi \in C_0^{2,4k}(\overline{D}_T)$ and integrating obtained equality over the domain D_T with taking into account that the functions from the space $C_0^{2,4k}(\overline{D}_T)$ satisfy the boundary conditions (2) and (3), we get

$$\int\limits_{D_T} \Big[\frac{\partial u}{\partial t}\,\frac{\partial \varphi}{\partial t} + \frac{\partial^{2k} u}{\partial x^{2k}}\,\frac{\partial^{2k} \varphi}{\partial x^{2k}}\Big]\,dx\,dt - \int\limits_{D_T} f(u)\varphi\,dx\,dt = -\int\limits_{D_T} F\varphi\,dx\,dt \ \ \forall\,\varphi \in C_0^{2,4k}(\overline{D}_T). \tag{5}$$

We take the equality (5) as a basis of definition of a weak generalized solution of the problem (1)–(3) in the space $W_0^{1,2k}(D_T)$. But for this, certain restrictions must be imposed on the function f so that the integral

$$\int\limits_{D_T} f(u)\varphi\,dx\,dt \tag{6}$$

exists.

Remark 2. Below, from function f in the equation (1) we require that

$$f \in C(\mathbb{R}), \ \ |f(u)| \le M_1 + M_2|u|^\alpha, \ \ \alpha = const > 1, \ \ u \in \mathbb{R}, \tag{7}$$

where $M_i = const \ge 0$, $i = 1, 2$. As it is known, since the dimension of the domain $D_T \subset \mathbb{R}^2$ equals two, the embedding operator

$$I : W_2^1(D_T) \to L_q(D_T)$$

is linear and compact operator for any fixed $q = const > 1$. At the same time the Nemitskii operator $N : L_q(D_T) \to L_2(D_T)$, acting by formula $Nu = f(u)$, where $u \in L_q(D_T)$, and function f satisfies the condition (7) is bounded and continuous, when $q \ge 2\alpha$. Therefore, if we take $q = 2\alpha$, then the operator

$$N_0 = NI : W_2^1(D_T) \to L_2(D_T)$$

will be continuous and compact. Hence, in particular, we have that if $u \in W_2^1(D_T)$, then $f(u) \in L_2(D_T)$ and from $u_n \to u$ in the space $W_2^1(D_T)$ it follows $f(u_n) \to f(u)$ in the space $L_2(D_T)$.

Definition 1. Let function f satisfy the condition (7) and $F \in L_2(D_T)$. A function $u \in W_0^{1,2k}(D_T)$ is named a weak generalized solution of the problem (1)–(3) if the integral equality (5) holds for any function $\varphi \in W_0^{1,2k}(D_T)$, i.e.,

$$\int\limits_{D_T} \Big[\frac{\partial u}{\partial t}\,\frac{\partial \varphi}{\partial t} + \frac{\partial^{2k} u}{\partial x^{2k}}\,\frac{\partial^{2k} \varphi}{\partial x^{2k}}\Big]\,dx\,dt - \int\limits_{D_T} f(u)\varphi\,dx\,dt = -\int\limits_{D_T} F\varphi\,dx\,dt \ \ \forall\,\varphi \in W_0^{1,2k}(D_T). \tag{8}$$

Note that due to Remark 2 the integral (6) in the left-hand side of the equality (8) is defined correctly since from $u \in W_0^{1,2k}(D_T)$ it follows that $f(u) \in L_2(D_T)$, and since $\varphi \in L_2(D_T)$, then $f(u)\varphi \in L_1(D_T)$.

It is easy to see that if a weak generalized solution u of the problem (1)–(3) in the sense of Definition 1 belongs to the class $C_0^{2,4k}(\overline{D}_T)$, then it is a classical solution to this problem.

In the space $C_0^{2,4k}(\overline{D}_T)$ together with the scalar product

$$(u, v)_0 = \int\limits_{D_T} \Big[uv + \frac{\partial u}{\partial t}\,\frac{\partial v}{\partial t} + \sum_{i=1}^{2k} \frac{\partial^i u}{\partial x^i}\,\frac{\partial^i v}{\partial x^i}\Big]\,dx\,dt \tag{9}$$

with the norm $\|\cdot\|_0 = \|\cdot\|_{W_0^{1,2k}(D_T)}$, defined by the right−hand side of the equality (4), let us consider the following scalar product

$$(u,v)_1 = \int\limits_{D_T} \Big[\frac{\partial u}{\partial t}\,\frac{\partial v}{\partial t} + \frac{\partial^{2k} u}{\partial x^{2k}}\,\frac{\partial^{2k} v}{\partial x^{2k}} \Big]\,dx\,dt \tag{10}$$

with the norm

$$\|u\|_1^2 = \int\limits_{D_T} \Big[\Big(\frac{\partial u}{\partial t}\Big)^2 + \Big(\frac{\partial^{2k} u}{\partial x^{2k}}\Big)^2 \Big]\,dx\,dt, \tag{11}$$

where $u,v \in C_0^{2,4k}(\overline{D}_T)$.

The following inequalities

$$c_1\|u\|_0 \le \|u\|_1 \le c_2\|u\|_0 \;\; \forall\, u \in C_0^{2,4k}(\overline{D}_T)$$

with positive constants c_1 and c_2, not dependent on u, are valid. Hence due to (9)–(11) it follows that if we complete the space $C_0^{2,4k}(\overline{D}_T)$ with respect to the norm (11), then we obtain the same Hilbert space $W_0^{1,2k}(D_T)$ with the equivalent scalar products (9) and (10). Using this circumstance, one can prove the unique solvability of the linear problem corresponding to (1)–(3), when $f = 0$, i.e. for any $F \in L_2(D_T)$ there exists a unique solution $u = L_0^{-1}F \in W_0^{1,2k}(D_T)$ to this problem, where the linear operator

$$L_0^{-1} : L_2(D_T) \to W_0^{1,2k}(D_T)$$

is continuous.

Remark 3. From the above reasoning it follows that when the conditions (7) are fulfilled, the nonlinear problem (1)–(3) is equivalently reduced to the functional equation

$$u = L_0^{-1}\,[f(u) - F] \tag{12}$$

in the Hilbert space $W_0^{1,2k}(D_T)$.

As noted below, if the nonlinear function f is not required to fulfill other conditions in addition to (7), then the problem (1)–(3) may not have a solution. At the same time, if the additional condition

$$\lim_{|u|\to\infty} \sup \frac{f(u)}{u} \le 0 \tag{13}$$

is satisfied, an a priori estimate is proved for the solution of the functional equation (12) in the space $W_0^{1,2k}(D_T)$, from which, taking into account Remarks 2 and 3, the existence of a solution to the equation (12) follows, and, consequently, of the problem (1)–(3) in the space $W_0^{1,2k}(D_T)$ in the sense of Definition 1. Thus, the following theorem holds.

Theorem 1. *Let the conditions* (7) *and* (13) *be fulfilled. Then for any* $F \in L_2(D_T)$ *the problem* (1)–(3) *has at least one weak generalized solution* u *in the space* $W_0^{1,2k}(D_T)$ *in the sense of Definition 1.*

In turns out that in the case of the problem (1)–(3), the monotonicity of the function f one can ensure uniqueness of its solution.

Theorem 2. *If the condition* (7) *is fulfilled and* f *is a non-strictly decreasing function, i.e.*

$$(f(y) - f(z))(y - z) \le 0 \;\; \forall\, y,z \in \mathbb{R}, \tag{14}$$

then for any $F \in L_2(D_T)$ *the problem* (1)–(3) *can not have more than one weak generalized solution in the space* $W_0^{1,2k}(D_T)$ *in the sense of Definition 1.*

From these theorems it follows the following theorem.

Theorem 3. *Let the conditions* (7), (13) *and* (14) *be fulfilled. Then for any* $F \in L_2(D_T)$ *the problem* (1)–(3) *has a unique weak generalized solution* u *in the space* $\overset{\circ}{W}{}^{1,2k}_{0}(D_T)$ *in the sense of Definition 1.*

As noted above, if no other conditions are imposed on the nonlinear function f in addition to the condition (7), then the problem (1)–(3) may not have a solution. Indeed, the following theorem holds.

Theorem 4. *Let the function* f *satisfy the conditions* (7) *and*

$$f(u) \le -|u|^{\gamma} \quad \forall\, u \in \mathbb{R}, \quad \gamma = const > 1, \tag{15}$$

and the function $F = \beta F_0$, *where* $F_0 \in L_2(D_T)$, $F_0 > 0$ *in the domain* D_T, $\beta = const > 0$. *Then there exists a number* $\beta_0 = \beta_0(F_0, \gamma)$ *such that for* $\beta > \beta_0$ *the problem* (1)–(3) *can not have a weak generalized solution in the space* $W^{1,2k}_{0}(D_T)$ *in the sense of Definition 1.*

It is easy to see that when the condition (15) is fulfilled, then the condition (13) is violated.

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