Approximate Solution of the Optimal Control Problem for a Parabolic Differential Inclusion with Fast-Oscillating Coefficients on Infinite Interval

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1 Introduction

The averaging method is a powerful tool for analyzing and solving optimal control problems, in particular for systems described by differential equations and inclusions with rapidly oscillating coefficients. It was originally developed and rigorously justified by Krylov and Bogolyubov for the approximate analysis of oscillating processes in non-linear mechanics, and then further refined for the control-related problems, see, e.g. a monograph by Plotnikov [10]. Motivated by the modern control engineering applications, the averaging method has been recenlty applied to the solution of optimal control problems for linear by control systems with rapidly oscillating coefficients on a finite interval [9], and on the semi-axis [8]. The approximate solutions of the optimal control problems for non-linear systems of differential inclusions with fast-oscillating parameters were investigated in [11] and [3], for the cases of a finite interval and on the semi-axis, respectively. The optimal control problem on the semi-axis for the Poisson equation with nonlocal boundary conditions was studied in [4]. Further applications of the averaging method for parabolic systems with fast-oscillating coefficients were considered in [5–7].

In the present paper, we use the averaging method for the investigation of the optimal control problem for nonlinear parabolic differential inclusion with fast-oscillating (w.r.t. time variable) coefficients on an infinite time interval. With this, we prove that the optimal control for the problem with averaged coefficients can be considered as "approximately" optimal for the original system.

2 Setting of the problem and the main results

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. In a cylinder $Q = (0,+\infty) \times \Omega$, we consider an initial boundary-value problem for a parabolic inclusion

$$\begin{cases} \dfrac{\partial y}{\partial t} \in Ay + f\Big(\dfrac{t}{\varepsilon}, y(t,x)\Big) + g(y)u, \quad (t,x) \in Q, \\ y\big|_{\partial\Omega} = 0, \\ y\big|_{t=0} = y_0(x). \end{cases} \tag{2.1}$$

Here $\varepsilon > 0$ is a small parameter, $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \operatorname{conv}(\mathbb{R})$ is a given multivalued mapping, $g : \mathbb{R} \to \mathbb{R}$, $q : \Omega \times \mathbb{R} \to \mathbb{R}$ are given real-valued mappings, A is an elliptic operator which can be defined by the rule:

$$Ay = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \Big(\frac{\partial y}{\partial x_i}\Big),$$

y is an unknown state function, u is an unknown control function, which are determined by requirements

$$u \in U \subseteq L^2(Q), \tag{2.2}$$

$$J(y,u) = \int\limits_Q e^{-\gamma t} q(x, y(t,x))\, dt\, dx + \alpha \int\limits_Q u^2(t,x)\, dt\, dx \longrightarrow \inf, \tag{2.3}$$

where γ, α are positive constants.

We consider the problem of finding an approximate solution of (2.1)–(2.3) by transition to the averaged coefficients. For this purpose, we assume that there exists $\overline{f} : \mathbb{R} \to \mathbb{R}$ such that uniformly w.r.t. $y \in \mathbb{R}$ there exists

$$\mathrm{dist}_H \left(\overline{f}(y), \frac{1}{T} \int\limits_0^T f(s,y)\, ds \right) \longrightarrow 0, \ \ T \to \infty, \tag{2.4}$$

where $\mathrm{dist}_H(A,B)$ is Hausdorff metric between sets A and B, and integral of multivalued map we consider in the sense of Aumann [1].

Let us consider the following optimal control problem

$$\begin{cases} \dfrac{\partial y}{\partial t} \in Ay + \overline{f}(y) + g(y)u, \quad (t,x) \in Q, \\ y\big|_{\partial\Omega} = 0, \\ y\big|_{t=0} = y_0(x), \end{cases} \tag{2.5}$$

$$u \in U \subseteq L^2(Q), \tag{2.6}$$

$$J(y,u) = \int\limits_Q e^{-\gamma t} q(x, y(t,x))\, dt\, dx + \alpha \int\limits_Q u^2(t,x)\, dt\, dx \longrightarrow \inf. \tag{2.7}$$

Under the natural assumptions on f, g, u, q we prove that the optimal control problem (2.1)–(2.3) has a solution $\{\overline{y}^{\varepsilon}, \overline{u}^{\varepsilon}\}$, i.e. for every $u \in U$ and for any solution y^{ε} of (2.1) with control u we have

$$J(\overline{y}^{\varepsilon}, \overline{u}^{\varepsilon}) \leq J(y^{\varepsilon}, u).$$

Note that we can apply similar suggestions to problem (2.5)–(2.7).

Assume that $\{\overline{y}, \overline{u}\}$ is a solution of (2.5)–(2.7). The main goal of the paper is to prove the convergence

$$J(\overline{y}^{\varepsilon}, \overline{u}^{\varepsilon}) \longrightarrow J(\overline{y}, \overline{u}), \ \ \varepsilon \to 0.$$

We suggest that the next assumptions for parameters of problem (2.1)–(2.3) are fulfilled.

Condition 2.1. Multi-valued function $f : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathrm{conv}(\mathbb{R})$ is continuous and there exist $C, C_1 > 0$ such that

$$\forall\, t \geq 0 \ \ \forall\, y \in \mathbb{R} \quad \|f(t,y)\|_+ := \sup_{\xi \in f(t,x)} \|\xi\|_{\mathbb{R}} \leq C + C_1 \|y\|_{\mathbb{R}},$$

where $\|\xi\|_{\mathbb{R}}$ denotes the Euclidian norm of $\xi \in \mathbb{R}^n$.

Condition 2.2. Function $g : \mathbb{R} \to \mathbb{R}$ is continuous function and there exists $C_2 > 0$ such that

$$\forall\, y \in \mathbb{R} \quad \|g(y)\|_{\mathbb{R}} \leq C_2.$$

Condition 2.3. Function $q : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and there exists $C_3 > 0$ and functions $K_1 \in L^2(\Omega)$, $K_2 \in L^1(\Omega)$ such that

$$\|q(x,\xi)\|_{\mathbb{R}} \leq C_3 \|\xi\|^2_{\mathbb{R}} + K_1(\Omega), \quad q(x,\xi) \leq K_2(x).$$

Condition 2.4. $U \subseteq L_2(Q)$ is closed and convex, $0 \in U$.

Condition 2.5. $\gamma > 2C_1^2 + 1 + C_2$.

Condition 2.6. Uniformly w.r.t. $y \in \mathbb{R}$ there exists the limit (2.4).

For $u \in U$ and $y_0 \in L^2(\Omega)$ we understand solution of (2.1) as a mild solution on every finite time interval, i.e. y is a solution of (2.1) if $y \in L^2_{loc}(0,+\infty; H^1_0(\Omega)) \cap L^\infty_{loc}(0,+\infty; L^2(\Omega))$ such that $\forall\, T > 0$, $\forall\, \varphi \in H^1_0(\Omega)$, $\forall\, \eta \in C^\infty_0(0,T)$ the following equality holds:

$$-\int_0^T (y,\varphi)_H \cdot \eta' \, dt + \int_0^T (\nabla y, \nabla \varphi)_H \cdot \eta \, dt = \int_0^T (f(t),\varphi)_H \cdot \eta \, dt + \int_0^T (g(y)u,\varphi)_H \cdot \eta \, dt, \quad f(t) \in f\Big(\frac{t}{\varepsilon}, y\Big)$$

and $f \in L^2_{loc}(0,+\infty; L^2(\Omega))$.

Here and after we denote by $\|\cdot\|_H$ and $(\cdot\,,\cdot)_H$ the classical norm and scalar product in $H = L^2(\Omega)$, by $\|\cdot\|_V$ the classical norm in $V := H^1_0(\Omega)$, by V^* the dual space to V.

Note that due to Conditions 2.1, 2.2 and properties of the operator A for y from definition of mild solution we have

$$\frac{\partial y}{\partial t} \in L^2_{loc}(0,+\infty; V^*).$$

In the sequel we denote by $\mathcal{F}^\varepsilon$ (or $\overline{\mathcal{F}}$) a set of all pairs $\{y,u\}$, where y is a solution of (2.1) (or (2.5)) with control u.

The following Lemma gives us a result on solvability of the optimal control problem (2.1)–(2.3).

Lemma. *Under Conditions 2.1–2.5 for every $\varepsilon > 0$ problem* (2.1)–(2.3) *has a solution $\{\overline{y}^\varepsilon, \overline{u}^\varepsilon\}$, that is*

$$J(\overline{y}^\varepsilon, \overline{u}^\varepsilon) \leq J(y,u) \quad \forall\, \{y,u\} \in \mathcal{F}^\varepsilon.$$

Note that the existence of a solution $\{\overline{y}, \overline{u}\}$ of (2.5)–(2.7) can be proved following similar arguments to the proof of the existence of $\{\overline{y}^\varepsilon, \overline{u}^\varepsilon\}$ for problem (2.1)–(2.3).

Theorem. *Suppose that Conditions 2.1-2.6 hold and, moreover, problem* (2.5) *has a unique solution for every $u \in U$.*

We assume additionally that $\forall\, \eta > 0\ \exists\, \delta > 0\ \forall\, t \geq 0\ \forall\, y,z \in \mathbb{R}$

$$\|y - z\|_{\mathbb{R}} < \delta \implies \operatorname{dist}(f(t,y), f(t,z)) < \eta.$$

Let $\{\overline{y}^\varepsilon, \overline{u}^\varepsilon\}$ be a solution of (2.1)–(2.3). *Then*

$$J(\overline{y}^\varepsilon, \overline{u}^\varepsilon) \longrightarrow J(\overline{y}, \overline{u}), \quad \varepsilon \to 0,$$

and up to subsequence

$$\overline{y}^\varepsilon \to \overline{y} \ \text{ in } L^2(0,+\infty; H),$$
$$\overline{u}^\varepsilon \to \overline{u} \ \text{ weakly in } L^2(0,+\infty; H),$$

where $\{\overline{y}, \overline{u}\}$ is a solution of (2.5)–(2.7).

These results are substantiated in [2].

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