An Efficient Numerical Method For Solving Problem for Impulsive Differential Equations with Loadings Subject to Multipoint Conditions

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1 Introduction

Impulsive differential equations are a significant area of mathematical research, driven by their ability to model real-world phenomena exhibiting sudden changes at specific moments. Such systems arise in various fields, including physics, biology, engineering, and economics, where abrupt transitions, discontinuities, or shocks are inherent. These equations offer a robust framework to capture behaviors like population explosions, mechanical shocks, or instantaneous changes in electrical circuits [1, 6, 8].

The concept of "loadings" in impulsive differential equations introduces an additional layer of complexity and applicability. Loadings can represent external influences or internal accumulations that act on the system during the impulse events [7]. This perspective extends the classical theory, enabling more comprehensive modeling of systems with cumulative or distributed effects accompanying the impulses.

The study of impulsive differential equations with loadings bridges the gap between theoretical advancements and practical applications. It explores existence, uniqueness, stability, and qualitative behavior of solutions while accounting for the dynamic interplay between impulses and loadings. Such investigations are critical in optimizing real-world systems, predicting outcomes, and controlling processes influenced by sudden changes and distributed forces [2, 5].

This paper focuses on developing numerical method for solving problem for impulsive differential equations with loadings subject to multipoint conditions. The objective is to provide numerical algorithm for solving problem for impulsive differential equations with loadings subject to multipoint conditions. By doing so, it contributes to the growing body of knowledge that supports both the theoretical understanding and practical use of impulsive systems in diverse scientific and engineering domains.

2 Setting of the problem and the main results

In this paper, by means of the Dzhumabaev parameterization method [3], we investigate the following problem for impulsive differential equations with loadings subject to multipoint conditions

$$\frac{dx}{dt} = A_0(t)x + \sum_{i=1}^{m} A_i(t) \lim_{t\to\theta_i+0} x(t) + f(t), \quad x \in \mathbb{R}^n, \quad t \in (0,T), \tag{2.1}$$

$$B_i \lim_{t\to\theta_i-0} x(t) - C_i \lim_{t\to\theta_i+0} x(t)$$

$$= \varphi_i + \sum_{k=1}^{i-1} D_k \lim_{t\to\theta_k-0} x(t) + \sum_{k=1}^{i-1} E_k \lim_{t\to\theta_k+0} x(t), \quad \varphi_i \in \mathbb{R}^n, \quad i = \overline{1,m}, \quad (2.2)$$

$$G_0 x(0) + G_1 \lim_{t\to\theta_1+0} x(t) + G_2 x(T) = d, \quad d \in \mathbb{R}^n. \quad (2.3)$$

Here $(n\times n)$-matrices $A_i(t)$ $(i = \overline{0,m})$ and n-vector-function $f(t)$ are piecewise continuous on $[0,T]$ with possible discontinuities of the first kind at the points $t = \theta_i$ $(i = \overline{1,m})$. B_i, C_i $(i = \overline{1,m})$, G_j $(j = \overline{0,2})$, D_k and E_k $(k = \overline{1,m-1})$ are constant $(n\times n)$-matrices, and φ_i $(i = \overline{1,m})$ and d are constant n vectors, $0 = \theta_0 < \theta_1 < \cdots < \theta_m < \theta_{m+1} = T$.

Let $PC([0,T],\theta,\mathbb{R}^n)$ denote the space of piecewise continuous functions $x(t)$ with the norm

$$\|x\|_1 = \max_{i=\overline{0,m}} \sup_{t\in[\theta_i,\theta_{i+1})} \|x(t)\|.$$

A solution to problem (2.1)–(2.3) is a piecewise continuously differentiable vector function $x(t)$ on $[0,T]$, which satisfies the system of the differential equations with loadings (2.1) on $[0,T]$ except the points $t = \theta_i$ $(i = \overline{1,m})$, the conditions of impulse effects at the fixed time points (2.2) and the condition (2.3).

Definition. Problem (2.1)–(2.3) is called uniquely solvable, if for any function $f(t) \in PC([0,T],\theta,\mathbb{R}^n)$ and vectors $d \in \mathbb{R}^n$, $\varphi_i \in \mathbb{R}^n$ $(i = \overline{1,m})$, it has a unique solution.

In this paper, we use the approach offered in [4] to solve the boundary value problem for impulsive differential equations with loadings subject to the multipoint conditions (2.1)–(2.3).

The interval $[0,T]$ is divided into subintervals by points:

$$[0,T) = \bigcup_{r=1}^{m+1} [\theta_{r-1},\theta_r).$$

Define the space $C([0,T],\theta,\mathbb{R}^{n(m+1)})$ of systems functions $x[t] = (x_1(t), x_2(t), \ldots, x_{m+1}(t))$, where $x_r : [\theta_{r-1},\theta_r) \to \mathbb{R}^n$ are continuous on $[\theta_{r-1},\theta_r)$ and have finite left-sided limits $\lim_{t\to\theta_r-0} x_r(t)$ for all $r = \overline{1,m+1}$, with the norm

$$\|x[\cdot]\|_2 = \max_{r=\overline{1,m+1}} \sup_{t\in[\theta_{r-1},\theta_r)} \|x_r(t)\|.$$

Denote by $x_r(t)$ the restriction of the function $x(t)$ to the $r-$th interval $[\theta_{r-1},\theta_r)$, i.e. $x_r(t) = x(t)$ for $t \in [\theta_{r-1},\theta_r)$, $r = \overline{1,m+1}$, and introducing the parameters

$$\lambda_r = \lim_{t\to\theta_{r-1}+0} x_r(t), \quad r = \overline{1,m+1},$$

and performing a replacement of the function $u_r(t) = x_r(t) - \lambda_r$ on each interval $[\theta_{r-1},\theta_r)$, $r = \overline{1,m+1}$, we obtain the boundary value problem with parameters λ_r, $r = \overline{1,m+1}$:

$$\frac{du_r}{dt} = A_0(t)(u_r + \lambda_r) + \sum_{i=1}^{m} A_i(t)\lambda_{i+1} + f(t), \quad t \in [\theta_{r-1},\theta_r), \quad r = \overline{1,m+1}, \quad (2.4)$$

$$u_r(\theta_{r-1}) = 0, \quad r = \overline{1,m+1}, \quad (2.5)$$

$$B_i \lim_{t\to\theta_i-0} u_i(t) + B_i\lambda_i - C_i\lambda_{i+1} = \varphi_i + \sum_{k=1}^{i-1} D_k \lim_{t\to\theta_k-0} \big[u_k(t) + \lambda_k\big] + \sum_{k=1}^{i-1} E_k\lambda_{k+1}, \quad i = \overline{1,m}, \quad (2.6)$$

$$G_0\lambda_1 + G_1\lambda_2 + G_2\lambda_{m+1} + G_2 \lim_{t\to T-0} u_{m+1}(t) = d. \tag{2.7}$$

A solution to problem (2.4)–(2.7) is a pair $(\lambda^*, u^*[t])$, with elements

$$\lambda^* = (\lambda_1^*, \lambda_2^*, \dots, \lambda_{m+1}^*) \in \mathbb{R}^{n(m+1)},$$
$$u^*[t] = \big(u_1^*(t), u_2^*(t), \dots, u_{m+1}^*(t)\big) \in C([0,T], \theta, \mathbb{R}^{n(m+1)}),$$

where $u_r^*(t)$ are continuously differentiable on $[\theta_{r-1}, \theta_r)$, $r = \overline{1, m+1}$, and satisfying the system of ordinary differential equations (2.4) and conditions (2.5)–(2.7) at $\lambda_r = \lambda_r^*$, $j = \overline{1, m+1}$.

Problem (2.1)–(2.3) is equivalent to problem (2.4)–(2.7). If the function $x^*(t)$ is a solution to problem (2.1)–(2.3), then the triple $(\lambda^*, u^*[t])$, where

$$\lambda^* = \big(x^*(\theta_0), x^*(\theta_1), \dots, x^*(\theta_m)\big)$$

and

$$u^*[t] = \big(x^*(t) - x^*(\theta_0), x^*(t) - x^*(\theta_1), \dots, x^*(t) - x^*(\theta_m)\big),$$

is a solution to problem (2.4)–(2.7). Conversely, if the triple $(\widetilde{\lambda}, \widetilde{u}[t])$, with elements

$$\widetilde{\lambda} = (\widetilde{\lambda}_1, \widetilde{\lambda}_2, \dots, \widetilde{\lambda}_{m+1}), \quad \widetilde{u}[t] = \big(\widetilde{u}_1(t), \widetilde{u}_2(t), \dots, \widetilde{u}_{m+1}(t)\big),$$

is a solution to problem (2.4)–(2.7), then the function $\widetilde{x}(t)$ defined by the equalities

$$\widetilde{x}(t) = \widetilde{u}_r(t) + \widetilde{\lambda}_r, \ \ t \in [\theta_{r-1}, \theta_r), \ \ r = \overline{1, m+1}$$

and

$$\widetilde{x}(T) = \widetilde{\lambda}_{m+1} + \lim_{t\to T-0} \widetilde{u}_{m+1}(t),$$

will be the solution of the original problem (2.1)–(2.3).

Let $\Phi_r(t)$ be a fundamental matrix to the differential equation

$$\frac{dx}{dt} = A(t)x \ \text{ on } [\theta_{r-1}, \theta_r], \ \ r = \overline{1, m+1}.$$

Then, the solution to the Cauchy problem (2.5), (2.6) can be written as follows

$$u_r(t) = \Phi_r(t) \int\limits_{\theta_{r-1}}^{t} \Phi_r^{-1}(\tau) A_0(\tau)\, d\tau\, \lambda_r + \Phi_r(t) \int\limits_{\theta_{r-1}}^{t} \Phi_r^{-1}(\tau) \sum_{i=1}^{m} A_i(\tau)\, d\tau\, \lambda_{i+1}$$
$$+ \Phi_r(t) \int\limits_{\theta_{r-1}}^{t} \Phi_r^{-1}(\tau) f(\tau)\, d\tau, \ \ t \in [\theta_{r-1}, \theta_r), \ \ r = \overline{1, m+1}. \tag{2.8}$$

Substituting the right-hand side of (2.8) into the impulse conditions (2.6) and condition (2.7) at the corresponding limit values, we obtain the following system of linear algebraic equations with respect to parameters λ_r, $r = \overline{1, m+1}$:

$$B_i \Phi_i(\theta_i) \int\limits_{\theta_{i-1}}^{\theta_i} \Phi_i^{-1}(\tau) \Big\{ A_0(\tau)\lambda_i + \sum_{j=1}^{m} A_j(\tau)\lambda_{j+1} \Big\}\, d\tau + B_i\lambda_i - C_i\lambda_{i+1}$$

$$-\sum_{k=1}^{i-1} D_k\lambda_k - \sum_{k=1}^{i-1} E_k\lambda_{k+1} - \sum_{k=1}^{i-1} D_k\Phi_k(\theta_k)\int\limits_{\theta_{k-1}}^{\theta_k} \Phi_k^{-1}(\tau)\Big\{A_0(\tau)\lambda_k + \sum_{j=1}^{m} A_j(\tau)\lambda_{j+1}\Big\}\,d\tau$$

$$= \varphi_i - B_i\Phi_i(\theta_i)\int\limits_{\theta_{i-1}}^{\theta_i} \Phi_i^{-1}(\tau)f(\tau)\,d\tau + \sum_{k=1}^{i-1} D_k\Phi_k(\theta_k)\int\limits_{\theta_{k-1}}^{\theta_k} \Phi_k^{-1}(\tau)f(\tau)\,d\tau, \quad i=\overline{1,m}, \tag{2.9}$$

$$G_0\lambda_1 + G_1\lambda_2 + G_2\Big[I + \Phi_{m+1}(T)\int\limits_{\theta_m}^{T} \Phi_{m+1}^{-1}(\tau)A_0(\tau)\,d\tau\Big]\lambda_{m+1}$$

$$+ G_2\Phi_{m+1}(T)\int\limits_{\theta_m}^{T} \Phi_{m+1}^{-1}(\tau)\sum_{j=1}^{m} A_j(\tau)\lambda_{j+1}\,d\tau = d - G_2\Phi_{m+1}(t)\int\limits_{\theta_m}^{T} \Phi_{m+1}^{-1}(\tau)f(\tau)\,d\tau. \tag{2.10}$$

We denote the matrix corresponding to the left side of the system of equations (2.9), (2.10) by $Q_*(\theta)$ and write the system in the form

$$Q_*(\theta)\lambda = F_*(\theta), \quad \lambda \in \mathbb{R}^{n(m+1)}, \tag{2.11}$$

where

$$F_*(\theta) = \begin{pmatrix} \varphi_1 - B_1\Phi_1(\theta_1)\int\limits_{\theta_0}^{\theta_1} \Phi_1^{-1}(\tau)f(\tau)\,d\tau \\ \varphi_2 - B_2\Phi_2(\theta_2)\int\limits_{\theta_1}^{\theta_2} \Phi_2^{-1}(\tau)f(\tau)\,d\tau + D_1\Phi_1(\theta_1)\int\limits_{\theta_0}^{\theta_1} \Phi_1^{-1}(\tau)f(\tau)\,d\tau \\ \vdots \\ \varphi_m - B_m\Phi_m(\theta_m)\int\limits_{\theta_{m-1}}^{\theta_m} \Phi_m^{-1}(\tau)f(\tau)\,d\tau + \sum_{k=1}^{m-1} D_k\Phi_k(\theta_k)\int\limits_{\theta_{k-1}}^{\theta_k} \Phi_k^{-1}(\tau)f(\tau)\,d\tau \\ d - \Phi_{m+1}(t)\int\limits_{\theta_m}^{T} \Phi_{m+1}^{-1}(\tau)f(\tau)\,d\tau \end{pmatrix}.$$

Theorem. *Let the matrix $Q_*(\theta) : \mathbb{R}^{n(m+1)} \to \mathbb{R}^{n(m+1)}$ be invertible. Then the boundary value problem (2.1)–(2.3) has a unique solution $x^*(t)$ for any $f(t) \in PC([0,T],\theta,\mathbb{R}^n)$, $d \in \mathbb{R}^n$, and $\varphi_i \in \mathbb{R}^n$, $i=\overline{1,m}$.*

Solvability of the boundary value problem (2.1)–(2.3) is equivalent to the solvability of system (2.11). The solution to system (2.11) is a vector λ^*, consisting of the values of solutions to problem (2.1)–(2.3) at the initial points of subintervals, i.e., $\lambda_r^* = x^*(\theta_{r-1})$, $r=\overline{1,m+1}$.

If $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_{m+1}^*)$ solution to system (2.11) is known, then a solution to problem (2.1)–(2.3) is determined by the equalities:

$$x^*(t) = \Phi_r(t)\Phi_r^{-1}(\theta_{r-1})\lambda_r^* + \Phi_r(t)\int\limits_{\theta_{r-1}}^{t} \Phi_r^{-1}(\tau)\sum_{j=1}^{m} A_j(\tau)\,d\tau\,\lambda_{j+1}^*$$

$$+ \Phi_r(t) \int\limits_{\theta_{r-1}}^{t} \Phi_r^{-1}(\tau) f(\tau)\, d\tau, \ \ t \in [\theta_{r-1}, \theta_r), \ \ r = \overline{1, m+1}, \tag{2.12}$$

$$x^*(T) = \Phi_{m+1}(t)\Phi_{m+1}^{-1}(\theta_m)\lambda_{m+1}^* + \Phi_{m+1}(t) \int\limits_{\theta_m}^{T} \Phi_{m+1}^{-1}(\tau) \sum_{j=1}^{m} A_j(\tau)\, d\tau\, \lambda_{j+1}^*$$
$$+ \Phi_{m+1}(t) \int\limits_{\theta_m}^{T} \Phi_{m+1}^{-1}(\tau) f(\tau)\, d\tau. \tag{2.13}$$

Expressions (2.12) and (2.13) give the analytical form of solution to problem (2.1)–(2.3).

We offer the following algorithm for numerical solving of linear boundary value problem for impulsive differential equations with loadings subject to the multipoint conditions (2.1)–(2.3).

1. Suppose we have a partition: $0 = \theta_0 < \theta_1 < \cdots < \theta_m < \theta_{m+1} = T$. Divide each rth interval $[\theta_{r-1}, \theta_r]$, $r = \overline{1, m+1}$, into N_r parts.

2. Solve the following Cauchy problem for ordinary differential equations

$$\frac{dz}{dt} = A_0(t)z + A_j(t), \ \ z(\theta_{r-1}) = 0, \ \ t \in [\theta_{r-1}, \theta_r], \ \ j = \overline{0, m}, \ \ r = \overline{1, m+1},$$
$$\frac{dz}{dt} = A_0(t)z + f(t), \ \ z(\theta_{r-1}) = 0, \ \ t \in [\theta_{r-1}, \theta_r], \ \ r = \overline{1, m+1}.$$

3. Construct the system of linear algebraic equations in parameters

$$Q_*^{\widetilde{h}}(\theta)\lambda = F_*^{\widetilde{h}}(\theta), \ \ \lambda \in \mathbb{R}^{n(m+1)},$$

and find its solution $\lambda^{\widetilde{h}}$. As noted above, the elements of $\lambda^{\widetilde{h}} = (\lambda_1^{\widetilde{h}}, \lambda_2^{\widetilde{h}}, \dots, \lambda_{m+1}^{\widetilde{h}})$ are the values of an approximate solution to problem (2.1)–(2.3) at the left end-points of the subintervals: $x^{\widetilde{h}_r}(\theta_{r-1}) = \lambda_r^{\widetilde{h}}$, $r = \overline{1, m+1}$.

4. To define the values of an approximate solution at the remaining points of set $\{\theta_{r-1}, \theta_r\}$, $r = \overline{1, m+1}$, we solve the Cauchy problems

$$\frac{dx}{dt} = A_0(t)x + \sum_{j=1}^{m} A_j(t)\lambda_{j+1}^{\widetilde{h}} + f(t), \quad x(\theta_{r-1}) = \lambda_r^{\widetilde{h}}, \ \ t \in [\theta_{r-1}, \theta_r], \ \ r = \overline{1, m+1}.$$

Thus, this algorithm allows us to find the numerical solution to problem (2.1)–(2.3).

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