Nonlinear Functional Integral Itô Equations: Existence and Uniqueness

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Nonlinear deterministic and stochastic integral equations of the Hammerstein type have a long history. These equations are known to play a major role in classical problems of physics and engineering. Due to the expansion of the scope of applications of integral equations, in particular, to problems in biology and mathematical economics, various generalizations of the Hammerstein equations are becoming increasingly popular in the literature.

We consider the following Hammerstein-type stochastic equation with singular and non-singular kernels and nonlinear Volterra operators:

$$x(t) = \kappa(t) + \sum_{i=1}^{m} \int_0^t K_i(t,s)(F_i x)(s)\, ds + \sum_{i=1}^{m} \sum_{j=1}^{m_i} \int_0^t K_{ij}(t,s)(G_{ij}x)(s)\, dB_i(s), \tag{1}$$

where $x(t)$, $\kappa(t)$ are random n-dimensional processes, B_i are jointly independent scalar Wiener processes, K_i, K_{ij} are deterministic Borel functions with values in the space of $n \times n$-matrices, and F_i and G_{ij} are Volterra operators ensuring the dependence of solutions of the equations on the prehistory. Here the first integral is the Lebesgue integral, and the second is the Itô integral. In most formulations below, equation (1) is assumed to be defined on a finite interval $[0,T]$, but in fact, all the results are also true for the semiaxis $t \geq 0$.

Equation (1) covers many important classes of stochastic fractional differential and integral equations. To see how a stochastically perturbed deterministic equation with fractional derivatives can be converted into (1), consider the deterministic equation

$$({}^C D_{0+}^{\alpha} x)(t) = f(t, x(t)) \ \ (\alpha > 0),$$

with the fractional Caputo derivative, see e.g. the monograph [6]. If this equation is perturbed by the white noise $\dot{B}(t)$, then we obtain a formally written equation

$$({}^C D_{0+}^{\alpha} x)(t) = f(t, x(t)) + g(t, x(t))\dot{B}(t)$$

or

$$d^{\alpha} x(t) = f(t, x(t))\, dt + g(t, x(t))\, dB(t), \tag{2}$$

where d^α is the fractional Caputo differential. In this case, the transition from (2) to a well-defined integral equation (1) is based on the fractional integration formula

$$({}^C D^\alpha_{0+} x)(t) = f(t) \implies x(t) = \sum_{k=0}^{l-1} \frac{x^{(k)}(0)t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

where $\Gamma(\alpha) = \int_0^\infty s^{\alpha-1} e^{-s}\, ds$ – the Gamma function, and $l = \alpha$ if $\alpha \in N$, and $l = [\alpha] + 1$ if $\alpha \notin N$. This formula allows us to move from the differential form (2) to the integral one:

$$x(t) = \sum_{k=0}^{l-1} \frac{x^{(k)}(0)t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, x(s))\, dB(s),$$

and then a solution of equation (2) is by definition understood as a stochastic process $x(t)$ satisfying this integral equation.

Another example of (1) are equations with fractional Wiener processes describing a popular class of models primarily developed in connection with their applications in financial mathematics, see, for example, [3], as well as numerous references cited in this monograph. An example is an equation of the form

$$dx(t) = f(t, x(t))\, dt + g(t, x(t))\, dB^\beta(t), \tag{3}$$

where B^β is a fractional Wiener process with the Hurst parameter β $(0.5 < \beta < 1)$. Note that without loss of generality we can assume that B^β is written in the Riemann–Liouville form, since this form differs from the standard one by a progressively measurable stochastic process with absolutely continuous trajectories, which can therefore be included in the first term on the right-hand side of the equation (3). This observation makes it possible to write the equation (3) as an integral equation (1) using the well-known formula [3]

$$\int_0^t \xi(t)\, dB^\beta(t) = \frac{1}{\Gamma(\beta + 1/2)} \int_0^t \xi(s)(t-s)^{\beta-1/2}\, dB(t).$$

Then equation (3) can be rewritten in the integral form

$$x(t) = x(0) + \int_0^t f(s, x(s))\, ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s, x(s))\, dB(s),$$

where $\alpha = \beta + 1/2$. By a solution of equation (3) we mean a stochastic process $x(t)$ satisfying this integral equation in order to avoid technical difficulties associated with integration over fractional Wiener process [3].

The third important class of stochastic equations included in the general form (1) are equations with multiple time scales

$$dx(t) = \sum_{i=1}^m f_i(t, x(t))\, (dt)^{\alpha_i} + g(t, x(t))\, dB(t) \ \ (0 < \alpha_i < 1), \tag{4}$$

which were introduced in [9]. Here $(dt)^{\alpha_i}$ are Jumarie-type differentials defining independent time scales $T_i(t) = t^{\alpha_i}$ (see [9] for a more detailed description of these time scales). The transition from

(4) to the integral equation (1) is based on the formula

$$\int_{t_0}^{t} \xi(t)(dt)^{\alpha} = \alpha \int_{t_0}^{t} \xi(s)(t-s)^{\alpha-1}\, dt,$$

developed in [9], which again gives a special case of the equation (1):

$$x(t) = x(0) + \sum_{i=1}^{m} \alpha_i \int_{0}^{t} (t-s)^{\alpha_1 - 1} f_i(s, x(s))\, ds + \int_{0}^{t} g(s, x(s))\, dB(s).$$

By combining all these special cases, one can also obtain various mixed integral equations (1) with singular kernels K_i, K_{ij} of the form $const(t-s)^{\alpha-1}$ $(0 < \alpha < 1)$, and some further examples can be found in Corollaries 1–6 below.

In what follows, we use the following constants that remain fixed:

- $n \in N$ is the dimension of the phase space of the equation, i.e. the size of the solution vector of the equation.
- $m, m_i \in N$.
- i is the index satisfying the conditions $1 \le i \le m$.
- j is the index satisfying the conditions $1 \le j \le m_i$.
- $T > 0$, $p \ge 2$, $q \ge 1$, $q_i \ge 1$, $q_{ij} \ge 1$, $\alpha_i > 0$, $\alpha_{ij} > 1/2$ – real numbers.

The following notations will also be used:

- $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_- = (-\infty, 0)$.
- $|\cdot|$ – fixed norm in $\mathbb{R}^n$ and $\|\cdot\|$ – matrix norm consistent with the norm $|\cdot|$.
- I_A – indicator (characteristic function) of the set A.
- Bor(M) – σ-algebra of all Borel subsets of the metric space M.
- L_q^n – Lebesgue space of equivalence classes of n-dimensional functions on the interval $[0, T]$.
- $\mathcal{B} = (\Omega, \mathcal{F}, (\mathcal{F})_{t \ge 0}, P)$ is a stochastic basis, where Ω is the set of elementary events, $\mathcal{F}$ is the σ-algebra of events on Ω, $(\mathcal{F})_{t \ge 0}$ is a right-continuous non-decreasing flow of σ-subalgebras of $\mathcal{F}$, P is a probability measure on $\mathcal{F}$, and all σ-algebras are complete with respect to this measure.
- E is the mathematical expectation constructed with respect to the measure P.
- $B(t)$ $(t \in \mathbb{R}_+)$ – scalar standard Wiener process.
- $B_i(t)$ $(t \in \mathbb{R}_+)$ – scalar standard and jointly independent Wiener processes.
- k_p^n – linear space of n-dimensional $\mathcal{F}_0$-measurable random variables χ satisfying the condition $E|\chi|^p < \infty$; the norm in k_p^n is the p-th root of this variable.
- $\mathcal{D}_p^n$ is the linear normed space of all n-dimensional progressively measurable stochastic processes $x(\cdot)$ on the interval $[0, T]$ satisfying the condition $\sup\limits_{0 \le t \le T} E|x(t)|^p < \infty$; the norm in $\mathcal{D}_p^n$ is the p-th root of this quantity.

Let J be the interval $[0,T]$ or the semiaxis $\mathbb{R}_+$. Recall that a stochastic process $x(t,\omega)$ ($t \in J$, $\omega \in \Omega$) whose restriction to the set $[0,v] \times \Omega$ is $\mathrm{Bor}([0,v]) \otimes \mathcal{F}_v$-measurable for any $v \in J$, is called progressively measurable (with respect to the stochastic basis $\mathcal{B}$).

Definition 1. Let a Volterra operator V map stochastic processes from $\mathcal{D}_p^n$ to progressively measurable processes and let there exist a linear bounded operator $Q : \mathcal{D}_p^n \to \mathcal{D}_p^n$ and a measurable deterministic function $\Psi(t) \geq 0$, $t \in [0,T]$, such that the inequality

$$\big|(Vx)(t) - (Vy)(t)\big| \leq \Psi(t)|(Q(x-y))(t)|$$

for all $x, y \in \mathcal{D}_p^n$ and μ-almost all $0 \leq t \leq T$. Then we will say that the operator V satisfies *the generalized Lipschitz condition with the operator Q and the function Ψ.*

The theorem below describes conditions of existence and uniqueness of the main equation (1).

Theorem 1. *Let the following conditions be satisfied for the equation* (1) *on the interval* $[0,T]$*:*

(1) $\kappa \in \mathcal{D}_p^n$.

(2) *The operators F_i, G_{ij} satisfy the generalized Lipschitz conditions with linear bounded operators $Q_i, Q_{ij} : \mathcal{D}_p^n \to \mathcal{D}_p^n$ and functions $\Psi_i \in L^1_{q_i}$, $\Psi_{ij} \in L^1_{2q_{ij}}$, respectively.*

(3) $F_i\widehat{0} \in \mathcal{D}_p^n$, $G_{ij}\widehat{0} \in \mathcal{D}_p^n$, *where $\widehat{0}$ is the zero element of $\mathcal{D}_p^n$.*

(4) $C_i := \sup\limits_{0 \leq t \leq T} \displaystyle\int_0^t \|K_i(t,s)\|^{\frac{q_i}{q_i-1}}\,ds < \infty$, $\quad C_{ij} := \sup\limits_{0 \leq t \leq T} \displaystyle\int_0^t \|K_{ij}(t,s)\|^{\frac{2q_{ij}}{q_{ij}-1}}\,ds < \infty.$

Then this equation has a unique solution, belonging to the space $\mathcal{D}_p^n$.

In what follows we apply Theorem 1 to several specific classes of stochastic fractional equations. The interval on which the existence of solutions is proved is always assumed to be finite and equal to $[0,T]$, and the solution on this interval belongs to the space $\mathcal{D}_p^n$, but all the results below remain valid for the semi-axis with obvious changes in the formulations.

Corollary 1. *Let in the equation* (1) $K_i(t,s) = (t-s)^{\alpha_i - 1}$, $K_{ij}(t,s) = (t-s)^{\alpha_{ij}-1}$, *and the operators F_i, G_{ij} satisfy conditions* (2), (3) *of Theorem 1, where*

$$q_i > \max\{\alpha_i^{-1}; 1\}, \quad q_{ij} > \max\{(2\alpha_{ij} - 1)^{-1}; 1\}.$$

Then for any $\kappa \in \mathcal{D}_p^n$ the equation (1) *has a unique solution.*

Corollary 1 is a far-reaching generalization of the corresponding results on fractional equations with Caputo derivatives from [8] (for the finite-dimensional case) and [4]. In particular, it includes random right-hand sides and random delays.

The two corollaries below deal with the initial value problem for equations with distributed and random delays, respectively. Both types of initial value problems are special cases of the equation (1), since, as shown below, they are reduced to this equation using the technique described in the monograph [2].

Consider the equation

$$\begin{aligned} x(t) = x(0) &+ \sum_{i=1}^m \int_0^t (t-s)^{\alpha_i - 1} f_i(s, (H_i x)(s))\,ds \\ &+ \sum_{i=1}^m \sum_{j=1}^{m_i} \int_0^t (t-s)^{\alpha_{ij}-1} g_{ij}(s, (H_{ij}x)(s))\,dB_i(s) \ \ (t \in [0,T]), \end{aligned} \tag{5}$$

where $f_i(t,\omega,v)$, $g_{ij}(t,\omega,v)$ – n-dimensional random functions that for each $v \in \mathbb{R}^{nl}$ are progressively measurable in variables $(t,\omega) \in [0,T] \times \Omega$, and for $P \otimes \mu$-almost all (t,ω) are continuous in v. The initial condition for (5) is defined by

$$x(s) = \varphi(s) \ \ (s \in \mathbb{R}_-), \tag{6}$$

where φ is a given stochastic process on $\mathbb{R}_-$. By a solution of the problem (5), (6) $x(t)$ $(t \le T)$ we mean an n-dimensional stochastic process whose restriction to the interval $[0,T]$ belongs to the space $\mathcal{D}_p^n$ and which satisfies the initial condition (6).

Let us start with an equation that includes distributed delay.

Corollary 2. *Let the following conditions be satisfied:*

(1) φ – $(\mathrm{Bor}(\mathbb{R}_-) \otimes \mathcal{F}_0)$*-measurable n-dimensional stochastic process.*

(2) *There exist measurable non-negative functions $\Psi_i(t)$, $\Psi_{ij}(t)$ $(t \in [0,T])$, $\Psi_i \in L^1_{q_i}$, $\Psi_{ij} \in L^1_{2q_{ij}}$, where $q_i > \max\{\alpha_i^{-1}; 1\}$, $q_{ij} > \max\{(2\alpha_{ij}-1)^{-1}; 1\}$, for which $P \otimes \mu$-the inequalities*

$$|f_i(t,u) - f_i(t,v))| \le \Psi_i(t)|u-v| \ \ \text{and} \ \ |g_{ij}(t,u) - g_{ij}(t,v))| \le \Psi_{ij}(t)|u-v|$$

for any $u, v \in \mathbb{R}^{nl}$ and $t \in [0,T]$.

(3)

$$(H_i z)(t) = \int_{-\infty}^{t} d_s\mathcal{R}_i(t,s)z(s), \quad (H_{ij} z)(t) = \int_{-\infty}^{t} d_s\mathcal{R}_{ij}(t,s)z(s),$$

where Borel functions $\mathcal{R}_i$, $\mathcal{R}_{ij}$, defined on the set $[0,T] \times (-\infty,t]$ and taking values in the space of $(nl) \times n$-matrices, satisfy conditions

$$\sup_{0\le t\le T} \mathrm{Var}_0^t\, \mathcal{R}_i(t,\,\cdot\,) < \infty, \quad \sup_{0\le t\le T} \mathrm{Var}_0^t\, \mathcal{R}_{ij}(t,\,\cdot\,) < \infty,$$

$$\sup_{0\le t\le T} E\left|f_i\Big(t, \int_{-\infty}^{0} d_s\mathcal{R}_i(t,s)\varphi(s)\Big)\right|^p < \infty, \quad \sup_{0\le t\le T} E\left|g_{ij}\Big(t, \int_{-\infty}^{0} d_s\mathcal{R}_{ij}(t,s)\varphi(s)\Big)\right|^p < \infty.$$

Then for any $x(0) \in k_p^n$ the problem (5), (6) has a unique solution.

The following corollary considers the initial value problem (5), (6) with random delays.

Corollary 3. *Let conditions* (1), (2) *of Corollary 2 be satisfied, and let condition* (3) *be replaced by condition*

3A. $(H_i z)(t) = (x(h_i^1(t)), \ldots, x(h_i^l(t)))$, $(H_{ij} z)(t) = (x(h_{ij}^1(t)), \ldots, x(h_{ij}^l(t)))$, *where scalar stochastic processes $h_i^k(t)$, $h_{ij}^k(t)$ $(k = 1,\ldots,l)$ satisfy the conditions $h(t) \le t$ a.s. $0 \le t \le T$, $h^{-1}(B) \in \mathrm{Bor}([0,T]) \otimes \mathcal{F}_v$ for any $v \in [0,T]$ and any Borel set $B \subset (-\infty, v]$ and*

$$\sup_{0\le t\le T} E\left|f_i\Big(t, \varphi(h_i^1(t))I_{\{h_i^1(t)<0\}}, \ldots, \varphi(h_i^l(t))I_{\{h_i^l(t)<0\}}\Big)\right|^p < \infty,$$

$$\sup_{0\le t\le T} E\left|g_{ij}\Big(t, \varphi(h_{ij}^1(t))I_{\{h_{ij}^1(t)<0\}}, \ldots, \varphi(h_{ij}^l(t))I_{\{h_{ij}^l(t)<0\}}\Big)\right|^p < \infty.$$

Then for any $x(0) \in k_p^n$ *equation with random delays*

$$x(t) = x(0) + \sum_{i=1}^{m} \frac{1}{\Gamma(\alpha_i)} \int_0^t (t-s)^{\alpha_i - 1} f_i\big(s, x(h_i^1(t)), \dots, x(h_i^l(t))\big)\, ds + \sum_{i=1}^{m} \sum_{j=1}^{m_i} \frac{1}{\Gamma(\alpha_{ij})} \int_0^t (t-s)^{\alpha_{ij}-1} g_{ij}\big(s, x(h_{ij}^1(t)), \dots, x(h_{ij}^l(t))\big)\, dB_i(s) \ \ (t \in [0,T])$$

has only one solution satisfying the equality (6)*.*

Consider now equations with arbitrary homogeneous singular kernels.

Let

$$x(t) = \kappa(t) + \sum_{i=1}^{m} \int_0^t K_i(t-s)(F_i x)(s)\, ds + \sum_{i=1}^{m} \sum_{j=1}^{m_i} \int_0^t K_{ij}(t-s)(G_{ij} x)(s)\, dB_i(s) \ \ (t \in [0,T]). \tag{7}$$

Corollary 4. *Let conditions* (1)–(3) *of Theorem 1 be satisfied, and condition* (4) *be replaced by*

4A. *The columns of the matrices* K_i *and* K_{ij} *belong to the spaces* $L^n_{q_i(q_i-1)^{-1}}$ *and* $L^n_{2q_{ij}(q_{ij}-1)^{-1}}$*, respectively.*

Then equation (7) *has a unique solution belonging to the space* $\mathcal{D}_p^n$*.*

Corollary 4 generalizes the main result of the paper [4].

Consider now equations including generalized fractional derivatives. They are represented by (1) on the interval $[0,T]$, where

$$K_i(t,s) = \psi_i'(s)(\psi_i(t) - \psi_i(s))^{\alpha_i - 1} \ \text{ and } \ K_{ij}(t,s) = \psi_{ij}'(s)(\psi_{ij}(t) - \psi_{ij}(s))^{\alpha_{ij}-1}, \tag{8}$$

the functions ψ_i and ψ_{ij} have continuous derivatives on $[0,T]$, and $\psi_i'(t) > 0$, $\psi_{ij}'(t) > 0$, $t \in [0,T]$. Obviously, this equation is a stochastic generalization of equations with Caputo derivatives.

Corollary 5. *Let the operators* F_i*,* G_{ij} *satisfy conditions* (2)*,* (3) *of Theorem 1, where*

$$q_i > \max\{\alpha_i^{-1}; 1\}, \quad q_{ij} > \max\big\{(2\alpha_{ij} - 1)^{-1}; 1\big\}.$$

Then for any $\kappa \in \mathcal{D}_p^n$ *the equation* (1)*, where* K_i *and* K_{ij} *are defined by the formulas* (8)*, has a unique solution.*

Corollary 5 generalizes the existence and uniqueness theorem from [1].

Finally, we consider equations including multifractional Wiener processes described by (1) on $[0,T]$, where

$$K_i(t,s) = \frac{1}{\Gamma(\theta_i(t))}(t-s)^{\theta_i(t)-1} \ \text{ and } \ K_{ij}(t,s) = c_{ij}(\theta_{ij}(t))(t-s)^{\theta_{ij}(t)-1/2}. \tag{9}$$

Corollary 6. *Let* $c_{ij}(u)$ $(u > 0)$*,* $\theta_i(t)$*,* $\theta_{ij}(t)$ $(t \in [0,T])$ *be Borel, bounded scalar functions, where* $\theta_i(t) \geq \alpha_i$*,* $\theta_{ij}(t) \geq \delta_{ij} > 0$ *for all* $t \in [0,T]$*. Let, further, the operators* F_i*,* G_{ij} *satisfy conditions* (2)*,* (3) *of Theorem 1, where*

$$q_i > \max\{\alpha_i^{-1}; 1\}, \quad q_{ij} > \max\big\{(2\delta_{ij})^{-1}; 1\big\}.$$

Then for any $\kappa \in \mathcal{D}_p^n$ *the equation* (1)*, where* K_i *and* K_{ij} *are defined by the formulas* (9)*, has a unique solution.*

Such equations were considered in [5]. Corollary 6 does not formally generalize the result on the existence of weak solutions for the equations offered in [5], but it does extend the existence and uniqueness theorem to equations of a much more general form.

The proofs of the above results can be found in the paper [7].

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