On the Dirichlet Type Problem for the Inhomogeneous Equation of String Oscillation

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Let Ω be a convex piecewise smooth domain on the plane of independent variables x and t. In the domain Ω for the inhomogeneous equation of the string vibration

$$\square\, u := u_{tt} - u_{xx} = F(x,t), \tag{1}$$

consider the Dirichlet type problem

$$u\big|_{\partial\Omega} = \varphi, \tag{2}$$

where $\square := \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$ and $\partial\Omega$ is the boundary of the domain Ω.

Numerous works are devoted to the investigation of Dirichlet type problem for the homogeneous string vibration equation. A. Sommerfeld [11] was the first who draw attention to this problem, pointing out the difference between this problem and Dirichlet problem for Laplace equation. A systematic study of this problem for the string vibration homogeneous equation dates back to the works of J. Adamard [5–7], which was later developed in the work of A. Huber [8]. Of particular note is the work of F. John [9], in which Dirichlet type problem for a fairly wide class of domains is reduced to the same problem for a rectangle. In the case of a rectangular domain, this problem was the subject of research of D. G. Bourgin and R. Duffin [2], N. N. Vakhania [12]. In these works, questions of uniqueness and existence of solutions are closely related to the algebraic properties of the ratio λ of sides of a rectangle. In particular, in the case when λ is irrational, there is a unique solution to Dirichlet problem. In the case when λ is rational, the uniqueness of the solution to this problem is violated, and some special cases of solvability of this problem are studied in the works of D. W. Fox and C. Pucci [4], L. L. Campbell [3].

In our work, although a special case is considered when $\lambda = 1$ for the inhomogeneous equation of forced oscillations of a string, necessary and sufficient conditions for the solvability of Dirichlet type problem with inhomogeneous boundary conditions are established, under which the solutions to this problem are written in quadratures. In particular, it is shown that the corresponding homogeneous problem have an infinite number of linearly independent solutions, which are given out explicitly.

Below, for simplicity and clarity of the obtained results, we will limit ourselves to considering the case, when the domain $\Omega := \{(x,t) \in \mathbb{R}^2 :\ 0 < x < l,\ 0 < t < l\}$ is a square. Rewrite the corresponding (2) boundary conditions on $\partial\Omega$ as follows

$$u(x,0) = \varphi(x), \quad u(x,l) = \varphi_1(x), \ \ 0 \le x \le l, \tag{3}$$

$$u(0,t) = \mu_1(t), \quad u(l,t) = \mu_2(t), \ \ 0 \le t \le l. \tag{4}$$

Considering regular solutions of the class $C^2(\overline{\Omega})$, we will require the following conditions of smoothness and consistency at the vertices of the square Ω to be satisfied for problem (1), (3), (4)

$$F \in C^1(\overline{\Omega}), \ \ \varphi,\ \varphi_1, \mu_i \in C^2([0,l]), \ \ i = 1,2,$$

$$\varphi(0)=\mu_1(0),\quad \varphi(l)=\mu_2(0),\quad \varphi_1(0)=\mu_1(l),\quad \varphi_1(l)=\mu_2(l),$$
$$\mu_1''(0)-\varphi''(0)=F(0,0),\quad \mu_1''(l)-\varphi_1''(0)=F(0,l),$$
$$\mu_2''(l)-\varphi_1''(l)=F(l,l),\quad \mu_2''(0)-\varphi''(l)=F(l,0).$$

Let $D:=PP_1P_3P_2$ be any characteristic rectangle, lying in Ω, where $P=P(x_0,t_0)$, $P_i=P_i(x_i,t_i)$, $t_0>t_i$, $i=1,2,3$, and the segments P_1P; P_3P_2 and P_1P_3; PP_2 belong to the families of characteristics $x-t=const$ and $x+t=const$, respectively.

Auxiliary statement. Let $\gamma=\gamma_1\cup\gamma_2$ be a simple piecewise smooth curve dividing the characteristic rectangle $PP_1P_3P_2$ into two simply connected domains D_1 and D_2, and γ_1 consists of the characteristic segments of equation (1), and γ_2 does not have a characteristic direction at any of its points. Next, let $u\in C^2(\overline{D}\setminus\gamma)\cap C(\overline{D})$ be a solution of equation (1) in $\overline{D}\setminus\gamma$, and the functions

$$u_1:=u\big|_{\overline{D}_1}\in C^2(\overline{D}_1)\ \text{ and }\ u_2:=u\big|_{\overline{D}_2}\in C^2(\overline{D}_2),$$

on the line transition γ are related by the following relations

$$u_1\big|_\gamma=u_2\big|_\gamma,\quad \frac{\partial u_1}{\partial\nu}\bigg|_{\gamma_2}=\frac{\partial u_2}{\partial\nu}\bigg|_{\gamma_2}, \tag{5}$$

where $\frac{\partial}{\partial\nu}$ is the derivative with respect to the direction of the outer unit normal $\nu:=(\nu_x,\nu_t)$ to the boundary one of the domains D_1 or D_2.

Then the equality holds

$$u(P)=u(P_1)+u(P_2)-u(P_3)+\frac{1}{2}\int\limits_{PP_1P_3P_2}F\,dx\,dt. \tag{6}$$

For investigation of the boundary value problem (1), (3), (4) below will be needed solution in quadratures of the following mixed problem: in the domain Ω find the solution $u\in C^2(\overline{\Omega})$ of equation (1) according to the initial

$$u(x,0)=\varphi(x),\quad u_t(x,0)=\psi(x),\ \ 0\le x\le l, \tag{7}$$

and boundary

$$u(0,t)=\mu_1(t),\quad u(l,t)=\mu_2(t),\quad 0\le t\le l, \tag{8}$$

conditions, where the functions $F,\ \varphi,\ \psi,\ \mu_1$ and μ_2 satisfy the following smoothness and consistency conditions

$$\begin{gathered}F\in C^1(\overline{\Omega}),\ \ \varphi\in C^2([0,l]),\ \ \psi\in C^1([0,l]),\ \ \mu_1,\ \mu_2\in C^2([0,l]),\\ F(0,0)=\mu_1''(0)-\varphi''(0),\quad F(l,0)=\mu_2''(0)-\varphi''(l),\\ \mu_1(0)=\varphi(0),\quad \mu_1'(0)=\psi(0),\quad \mu_2(0)=\varphi(l),\quad \mu_2'(0)=\psi(l).\end{gathered} \tag{9}$$

In order to solve these problem in quadratures let us divide the domain Ω, which is a square with vertices at the points $A(0,0)$, $B(0,l)$, $C(l,l)$ and $D(l,0)$, into four rectangular triangles $\Omega_1:=\Delta AOD$, $\Omega_2:=\Delta AOB$, $\Omega_3:=\Delta DOC$ and $\Omega_4:=\Delta BOC$, where the point $O(\frac{l}{2},\frac{l}{2})$ is the center of the square Ω (see, for example, [10]).

By virtue of d'Alembert's formula (see, for example, [1]) the solution of problem (1), (7) is given by the following equality

$$u(x,t)=\frac{1}{2}\big[\varphi(x-t)+\varphi(x+t)\big]+\frac{1}{2}\int\limits_{x-t}^{x+t}\psi(\tau)\,d\tau+\frac{1}{2}\int\limits_{\Omega^1_{x,t}}F\,d\xi\,d\tau,\quad (x,t)\in\Omega_1, \tag{10}$$

where $D^1_{x,t}$ is a triangle with vertices at the points (x,t), $(x-t,0)$ and $(x+t,0)$.

Let the point $P=P(x,t)\in\Omega_2$, and $PP_1P_3P_2$ be the characteristic rectangle, where $P_1=P_1(0,t-x)$, $P_2=P_2(x+t,0)$, $P_3=P_3(t,-x)$. Let us denote by $\widetilde{P}_2=\widetilde{P}_2(x-t,0)$ the point of intersection of the side P_1P_3 of the rectangle $PP_1P_3P_2$ with the side AD of the square Ω. For $t<0$, we introduce the function u_2 as a solution to the equation $\Box u_2=0$ with the initial conditions (7), i.e.,

$$u_2(x,t)=\frac{1}{2}\big[\varphi(x-t)+\varphi(x+t)\big]+\frac{1}{2}\int\limits_{x-t}^{x+t}\psi(\tau)\,d\tau,\ \ (x,t)\in\Delta AED, \tag{11}$$

where $E=E(\frac{l}{2},-\frac{l}{2})$. For the characteristic rectangle $PP_1P_3P_2$ we use equality (6), in which $\gamma=\widetilde{P}_2P_2$, $D_1=PP_1\widetilde{P}_2P_2$, $D_2=\widetilde{P}_2P_3P_2$, $u_1:=u\big|_{PP_1\widetilde{P}_2P_2}$, and the function u_2 is given by equality (11). Due to the above reasoning, bonding conditions (5) will be satisfied, and then, taking into account the Dirichlet boundary conditions (8), equality (6) for our case will take the form

$$u(x,t)=\mu_1(t-x)+\frac{1}{2}\big[\varphi(t+x)-\varphi(t-x)\big]+\frac{1}{2}\int\limits_{t-x}^{t+x}\psi(\tau)\,d\tau+\frac{1}{2}\int\limits_{PP_1\widetilde{P}_2P_2}F\,d\xi\,d\tau,\ \ (x,t)\in\Omega_2. \tag{12}$$

Carrying out similar reasoning in the case of $P=P(x,t)\in\Omega_3$ and $P=P(x,t)\in\Omega_4$ for solution $u=u(x,t)$ of problem (1), (7), (8) we will have

$$u(x,t)=\mu_2(x+t-l)+\frac{1}{2}\big[\varphi(x-t)-\varphi(2l-x-t)\big]+\frac{1}{2}\int\limits_{x-t}^{2l-x-t}\psi(\tau)\,d\tau+\frac{1}{2}\int\limits_{D^3_{x,t}}F\,d\xi\,d\tau,\ \ (x,t)\in\Omega_3, \tag{13}$$

and

$$\begin{aligned}u(x,t)&=\mu_1(t-x)+\mu_2(x+t-l)\\&\quad-\frac{1}{2}\big[\varphi(t-x)+\varphi(2l-t-x)\big]+\frac{1}{2}\int\limits_{t-x}^{2l-t-x}\psi(\tau)\,d\tau+\frac{1}{2}\int\limits_{D^4_{x,t}}F\,d\xi\,d\tau,\ \ (x,t)\in\Omega_4,\end{aligned} \tag{14}$$

respectively.

Here $D^3_{x,t}$ – quadrilateral with vertices: $P(x,t)\in\Omega_3$, $P^3_1(x-t,0)$, $P^3_2(2l-x-t,0)$ and $P^3_3(l,x+t-l)$, and $D^4_{x,t}$ – pentagon with vertices: $P(x,t)\in\Omega_4$, $P^4_1(0,t-x)$, $P^4_2(t-x,0)$, $P^4_3(2l-x-t,0)$ and $P^4_4(l,x+t-l)$.

Thus, due to the conditions of smoothness and consistency (9), the unique classical solution $u\in C^2(\overline{\Omega})$ of problem (1), (7), (8) is given by formulas (10), (12)–(14).

From the above reasoning the following theorem follows.

Theorem. *Let the smoothness and consistency conditions* (9) *be satisfied. Then for the solvability of Dirichlet problem* (1), (3), (4) *it is necessary and sufficient the following condition*

$$\varphi_1(x)=\mu_1(l-x)+\mu_2(x)-\varphi(l-x)+\frac{1}{2}\int\limits_{PP_1P_3P_2}F\,d\xi\,d\tau,\quad 0\le x\le l \tag{15}$$

to be satisfied, where

$$P=P(x,l),\ \ P_1=P_1(0,l-x),\ \ P_3=P_3(l-x,0),\ \ P_2=P_2(l,x).$$

Moreover, if condition (15) *is satisfied, all solutions to this problem are given by formulas* (10), (12)–(14), *where* ψ *is an arbitrary function from the class* $C^1([0,l])$.

From this theorem it follows that the kernel

$$K := \Big\{ v \in C^2(\overline{\Omega}),\ \ \square v = 0,\ \ v\big|_{\partial\Omega} = 0 \Big\}$$

of problem (1), (3), (4) *is infinite-dimensional and can be described by the formula*

$$v(x,t) = \begin{cases} \dfrac{1}{2} \displaystyle\int\limits_{x-t}^{x+t} \psi(\tau)\,d\tau, & (x,t) \in \Omega, \\ \dfrac{1}{2} \displaystyle\int\limits_{t-x}^{t+x} \psi(\tau)\,d\tau, & (x,t) \in \Omega, \\ \dfrac{1}{2} \displaystyle\int\limits_{x-t}^{2l-x-t} \psi(\tau)\,d\tau, & (x,t) \in \Omega, \\ \dfrac{1}{2} \displaystyle\int\limits_{t-x}^{2l-t-x} \psi(\tau)\,d\tau, & (x,t) \in \Omega_4, \end{cases} \tag{16}$$

where ψ *is an arbitrary function of the class* $C^1([0,l])$.

Remark. Taking into account that Dirichlet type problem (1), (3), (4) for the string vibration inhomogeneous equation, as well as Dirichlet problem for Poisson equation $\Delta u = F$ is self-adjoint, then a necessary condition for the solvability of problem (1), (3), (4) in the case of homogeneous boundary conditions (3), (4) is the following equality

$$\int\limits_{\Omega} Fv\,dx\,dt = 0 \ \ \forall\, v \in K,$$

where the function v is the given by equality (16).

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