Oscillation Theory of Nonlinear Differential Equations of Emden–Fowler Type with Variable Exponents

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1 Introduction

We consider nonlinear differential equations of the form

$$\big(p(t)\varphi_{\alpha(t)}(x')\big)' + q(t)\varphi_{\beta(t)}(x) = 0, \tag{A}$$

under the following assumptions:

(a) the coefficients $p(t)$ and $q(t)$ are positive continuous functions on $I = [a, \infty)$, $a \geq 0$;

(b) the exponents $\alpha(t)$ and $\beta(t)$ are positive continuous functions on I having the limits $\alpha(\infty)$ and $\beta(\infty)$ as $t \to \infty$ in the extended real number system;

(c) the symbol $\varphi_{\gamma(t)}$ with a positive continuous function $\gamma(t)$ on I denotes the operator in $C(I)$ defined by

$$\varphi_{\gamma(t)}(u(t)) = |u(t)|^{\gamma(t)} \operatorname{sgn} u(t), \quad u \in C(I).$$

Since the prototype of (A) is the differential equation

$$\big(p(t)\varphi_{\alpha}(x')\big)' + q(t)\varphi_{\beta}(x) = 0, \tag{A_0}$$

α and β being positive constants, which is well-known as the Emden–Fowler equation, (A) is often referred to as a generalized Emden–Fowler equation or an Emden–Fowler type equation with variable exponents.

We are concerned exclusively with nontrivial solutions $x(t)$ of (A) which are defined on an infinite interval of the form $[T, \infty)$, $T \geq a$. A solution is called *oscillatory* if it has an infinite sequence of zeros tending to infinity and *nonoscillatory* otherwise. Given a solution $x(t)$ of (A), we define

$$D_\alpha x(t) = p(t)\varphi_{\alpha(t)}(x'(t)),$$

and call it the *quasi-derivative* of $x(t)$. In this notation, the dependence of the operator D_α on $p(t)$ is omitted for simplicity.

Historically, a vast literature has been published on oscillation theory of the standard Emden–Fowler differential equation (A_0). A remarkable result in the theory is the fact that the situation in which all solutions of (A_0) with $\alpha \neq \beta$ are oscillatory can be characterized completely by the impressive integral conditions formulated in terms of the exponents $\{\alpha,\beta\}$ and the coefficients $\{p(t),q(t)\}$.

Equation (A_0) is called strongly superlinear or strongly sublinear according as $\alpha<\beta$ or $\alpha>\beta$, respectively. Use is made of the following notations and functions:

$$I_p=\int\limits_a^\infty p(t)^{-\frac{1}{\alpha}}\,dt,\quad I_q=\int\limits_a^\infty q(t)\,dt,$$

$$P(t)=\int\limits_a^t p(s)^{-\frac{1}{\alpha}}\,ds\ \text{ if }\ I_p=\infty,\quad \pi(t)=\int\limits_t^\infty p(s)^{-\frac{1}{\alpha}}\,ds\ \text{ if }\ I_p<\infty,$$

$$Q(t)=\int\limits_a^t q(s)\,ds\ \text{ if }\ I_q=\infty,\quad \rho(t)=\int\limits_t^\infty q(s)\ \text{ if }\ I_q<\infty.$$

The following facts are well-known.

(i) All solutions of (A_0) are oscillatory if $I_p=I_q=\infty$.

(ii) Assume that $I_p=\infty$ and $I_q<\infty$. Let (A_0) be strongly superlinear. All of its solutions are oscillatory if and only if

$$\int\limits_a^\infty \big(p(t)^{-1}\rho(t)\big)^{\frac{1}{\alpha}}\,dt=\infty.$$

(iii) Assume that $I_p=\infty$ and $I_q<\infty$. Let (A_0) be strongly sublinear. All of its solutions are oscillatory if and only if

$$\int\limits_a^\infty q(t)P(t)^\beta\,dt=\infty.$$

(iv) Assume that $I_p<\infty$ and $I_q=\infty$. Let (A_0) be strongly superlinear. All of its solutions are oscillatory if and only if

$$\int\limits_a^\infty q(t)\pi(t)^\beta\,dt=\infty.$$

(v) Assume that $I_p<\infty$ and $I_q=\infty$. Let (A_0) be strongly sublinear. All of its solutions are oscillatory if and only if

$$\int\limits_a^\infty \big(p(t)^{-1}Q(t)\big)^{\frac{1}{\alpha}}\,dt=\infty.$$

For the proofs of these theorems see e.g. Elbert and Kusano [1] and Kusano et al. [3].

Now, a question naturally arises: Is it possible to characterize the oscillation of all solutions of the generalized Emden–Fowler equations with variable exponents? The aim of the present work is to give an affirmative answer to this question by showing that the results (ii)–(v) for (A_0) mentioned above can be properly generalized to equation (A) which is *strongly superlinear* or *strongly sublinear* in the sense defined below.

Any generalized Emden–Fowler equation (A) is made up by the two crucial components. One is the pair of the exponents $\{\alpha(t),\beta(t)\}$ which determines the nonlinearity of (A), and the other is the pair of the coefficients $\{p(t),q(t)\}$ which implies, so to speak, the size or magnitude of (A).

The concept of superlinearity and sublinearity of (A_0) is extended to equation (A) as follows.

Definition 1.1.

(i) Equation (A) is said to be *strongly superlinear* if the pair of exponents $\{\alpha(t), \beta(t)\}$ has the property that $\alpha(t)$ is nonincreasing, $\beta(t)$ is nondecreasing and there is a constant $\lambda > 1$ such that

$$\beta(t) \geq \lambda\alpha(t) \ \text{ for } \ t \geq a.$$

(ii) Equation (A) is said to be *strongly sublinear* if the pair of exponents $\{\alpha(t), \beta(t)\}$ has the property that $\alpha(t)$ is nonincreasing, $\beta(t)$ is nondecreasing and there is a positive constant $\mu > 1$ such that

$$\alpha(t) \geq \mu\beta(t) \ \text{ for } \ t \geq a.$$

We measure the size of the coefficients $p(t)$ and $q(t)$ by their integrals defined by

$$I(p) = \int_a^\infty p(t)^{-\frac{1}{\alpha(t)}}\, dt \ \text{ and } \ I(q) = \int_a^\infty q(t)\, dt.$$

There are four different combinations of $I(p)$ and $I(q)$, of which the following three cases will be the main object of our analysis.

Definition 1.2. Equation (A) is said to be of category I if $I(p) = \infty$ and $I(q) < \infty$, of category II if $I(p) < \infty$ and $I(q) = \infty$, and of category III if $I(p) = \infty$ and $I(q) = \infty$.

The category IV ($I(p) < \infty$, $I(q) < \infty$) is excluded from our consideration because equation (A) of this category always possesses nonoscillatory solutions.

Our main objective in this paper is to generalize the propositions (ii)–(v) listed above regarding the standard Emden–Fowler equation (A_0) to the corresponding Emden–Fowler equation with variable exponents (A).

In Section 2 we focus our attention on equation (A) of category I and show by way of direct asymptotic analysis that necessary and sufficient conditions for oscillation of all of its solutions can be established for both strongly superlinear and strongly sublinear cases. Equation (A) of category II is considered in Section 3. There, we avoid analyzing the equation directly as in Section 2, and make use of an uncommon means named *Duality Principle* which makes it possible to derive the desired oscillation theorems for the category II equation almost automatically from the results on the category I equation already known in Section 2. Thus it turns out that our results obtained in Sections 2 and 3 combined are an exact generalization of the propositions (ii)–(v) which are the typical oscillation theorems for the standard Emden–Fowler equation (A_0).

2 Oscillation of equation (A) of category I

We begin with an oscillation theorem which generalizes the proposition (i) for (A_0) to equation (A) of of category III.

Theorem 2.1. *Consider equation* (A) *with* $\alpha(\infty) > 0$ *and* $\beta(\infty) > 0$. *All of its solutions are oscillatory if* $p(t)$ *and* $q(t)$ *have the property that* $I(p) = \infty$ *and* $I(q) = \infty$.

Proof. Assume for contradiction that (A) has a nonoscillatory solution $x(t)$ on $J = [T, \infty)$. Without loss of generality we may suppose that $x(t) > 0$ on J. Since (A) is written as

$$(D_\alpha x)'(t) = -q(t)x(t)^{\beta(t)} < 0,$$

$D_\alpha x(t)$ is decreasing on J. We claim that $D_\alpha x(t) > 0$ on J. In fact, if it is negative at some point of $t_* \in J$ then there is a negative constant $-k = D_\alpha x(t_*)$ such that

$$D_\alpha x(t) = -p(t)(-x'(t))^{\alpha(t)} \le -k \ \text{ for } \ t \ge t_*.$$

Rewriting the above as

$$-x'(t) \ge k^{\frac{1}{\alpha(t)}} p(t)^{-\frac{1}{\alpha(t)}} \ \text{ for } \ t \ge t_*,$$

and integrating the above inequality from t_* to t, we have

$$x(t_*) - x(t) \ge \int_{t_*}^{t} k^{\frac{1}{\alpha(s)}} p(s)^{-\frac{1}{\alpha(s)}}\, ds, \ \ t \ge t_*,$$

from which, since $k^{\frac{1}{\alpha(t)}} \ge k_0$, $t \ge t_*$, for some constant $k_0 > 0$ because of $\alpha(\infty) > 0$, it follows that $x(t) \to -\infty$ as $t \to \infty$. This, however, contradicts the assumed positivity of $x(t)$, and hence we must have $D_\alpha x(t) > 0$ on J. This means that $x(t)$ is increasing on J.

Now, we integrate (A) from T to t to obtain

$$\int_{T}^{t} q(s) x(s)^{\beta(s)}\, ds = D_\alpha x(T) - D_\alpha x(t) \le D_\alpha x(T), \ \ t \ge T,$$

which implies that $\int_T^\infty q(s)x(s)^{\beta(s)}\, ds < \infty$. Combining this inequality with the fact that $x(t)^{\beta(t)}$ with $\beta(\infty) > 0$ is greater than some positive constant on J, we conclude that $\int_T^\infty q(s)\, ds < \infty$ contrary to the assumption $I(q) = \infty$. This completes the proof. □

Note that in Theorem 2.1 neither the superlinearity nor the sublinearity is required for (A).

Let there be given equation (A) of category I whose coefficients $p(t)$ and $q(t)$ satisfy $I(p) = \infty$ and $I(q) < \infty$, respectively. Use is made of the functions

$$P_\alpha(t) = \int_{a}^{t} p(s)^{-\frac{1}{\alpha(s)}}\, ds \ \text{ and } \ \rho(t) = \int_{t}^{\infty} q(s)\, ds.$$

It is clear that $P_\alpha(t) \to \infty$ and $\rho(t) \to 0$ as $t \to \infty$.

The main results of this section are stated in the following two theorems. They guarantee that the situation in which all solutions of equation (A) of category I are oscillatory is completely characterized provided that (A) is either strongly superlinear or strongly sublinear.

Theorem 2.2. *Let equation* (A) *with* $\alpha(\infty) > 0$ *be of category* I *and strongly superlinear. Then, all solutions of* (A) *are oscillatory if and only if*

$$\int_{a}^{\infty} \big(p(t)^{-1} \rho(t)\big)^{\frac{1}{\alpha(t)}}\, dt = \infty. \tag{2.1}$$

Theorem 2.3. *Let equation* (A) *with* $\alpha(\infty) > 0$ *be of category* I *and strongly sublinear. Then, all solutions of* (A) *are oscillatory if and only if*

$$\int_{a}^{\infty} q(t) P_\alpha(t)^{\beta(t)}\, dt = \infty.$$

Each of these theorems is proved by reductio ad absurdum. In proving Theorem 2.2, for example, to verify the "if" part, first we assume (2.1) to hold but (A) has a nonoscillatory solution of (A) and after a sensitive computational process we are finally forced to admit the contrary conclusion that

$$\int_a^\infty \left(p(t)^{-1}\rho(t)\right)^{\frac{1}{\alpha(t)}}\,dt < \infty. \tag{2.2}$$

Likewise, to verify the "only if" part of Theorem 2.2, we have to show that the condition (2.2) implies the existence of a nonoscillatory solution for equation (A). As a matter of fact, one such positive solution $x(t)$ such that $x(\infty) = 1$ can be obtained as a solution of the integral equation with variable exponents

$$x(t) = 1 - \int_T^t \left(p(s)^{-1}\int_s^\infty q(r)x(r)^{\beta(r)}\,dr\right)^{\frac{1}{\alpha(s)}} ds, \quad t \geq T, \tag{2.3}$$

for some sufficiently large $T > a$. It should be noted that the solvability of (2.3) is assured for a much wider class of equations of the form (A) including both strongly superlinear and sublinear equations as special cases.

The procedure of the proof of Theorem 2.3 by reductio ad absurdum is essentially the same as for Theorem 2.2.

What is said above suggests that in studying oscillation theory of generalized Emden–Fowler equations preliminary knowledge of nonoscillation theory for them is indispensable. See [2].

3 Oscillation of equation (A) of category II via Duality Principle

Now we turn our attention to equation (A) of category II whose coefficients $p(t)$ and $q(t)$ satisfy the integral conditions

$$\int_a^\infty p(t)^{-\frac{1}{\alpha(t)}}\,dt < \infty, \quad \int_a^\infty q(t)\,dt = \infty.$$

Equation (A) is assumed to be either strongly superlinear or strongly sublinear.

For such an equation (A) the functions

$$\pi_\alpha(t) = \int_t^\infty p(s)^{-\frac{1}{\alpha(s)}}\,ds \text{ and } Q(t) = \int_a^t q(s)\,ds,$$

are well-defined and play a major role throughtout this section. It is clear that $\pi_\alpha(t) \to 0$ and $Q(t) \to \infty$ as $t \to \infty$.

Our aim is to find explicit oscillation criteria for equation (A) of category II which are similar to those given in Theorems 2.1 and 2.2 for equation (A) of category I. We are so bold as to make use of an uncommon method (named *Duality Principle*) which enables us to precisely formulate the desired results for equations of category II almost automatically (without additional serious computations) from the corresponding known results for equations of category I.

Let (A) be a generalized Emden–Fowler equation with the exponents $\{\alpha(t), \beta(t)\}$ and the coefficients $\{p(t), q(t)\}$. Putting

$$y(t) = -p(t)\varphi_{\alpha(t)}(x'(t)),$$

equation (A) is split into the first-order differential system with variable exponents

$$x'(t) = -p(t)^{-\frac{1}{\alpha(t)}} \varphi_{\frac{1}{\alpha(t)}}(y(t)), \quad y'(t) = q(t)\varphi_{\beta(t)}(x(t)). \tag{3.1}$$

It is easy to see that elimination of $\{y(t), y'(t)\}$ from (3.1) gives the original second-order differential equation (A), and that elimination of $\{x(t), x'(t)\}$ from (3.1) gives a new second-order differential equation

$$\big(q(t)^{-\frac{1}{\beta(t)}} \varphi_{\frac{1}{\beta(t)}}(y')\big)' + p(t)^{-\frac{1}{\alpha(t)}} \varphi_{\frac{1}{\alpha(t)}}(y) = 0. \tag{B}$$

Equation (B) is called the *reciprocal equation* of (A). Equation (B) is structurally the same as equation (A) and has the exponents $\{\frac{1}{\beta(t)}, \frac{1}{\alpha(t)}\}$ and the coefficients $\{q(t)^{-\frac{1}{\beta(t)}}, p(t)^{-\frac{1}{\alpha(t)}}\}$. It is obvious that (A) is the reciprocal equation of (B).

If we denote the exponents of (B) by $\{\widetilde{\alpha}(t), \widetilde{\beta}(t)\}$, and the coefficients of (B) by $\{\widetilde{p}(t), \widetilde{q}(t)\}$, then it is easily verified that the nonlinearity of (B) is the same as that of (A), and that

$$\int\limits_a^\infty \widetilde{p}(s)^{-\frac{1}{\widetilde{\alpha}(s)}}\, ds = \int\limits_a^\infty q(s)\, ds, \quad \int\limits_a^\infty \widetilde{q}(s)\, ds = \int\limits_a^\infty p(s)^{-\frac{1}{\alpha(s)}}\, ds.$$

Thus it is confirmed that the transition from equation (A) to its reciprocal equation (B) keeps the strong superlinearity or strong sublinearity of (A) unchanged, but changes the category of (A) from I to II, or from II to I. Such a close interrelationship between (A) and its reciprocal equation (B) is worthy of being remembered as a principle:

Duality Principle. Let equation (B) be the reciprocal equation of (A).

(i) If (A) is strongly superlinear (or strongly sublinear), then so is (B).

(ii) If (A) is of category I (resp. category II), then (B) is of category II (resp. category I).

(iii) All solutions of (A) are oscillatory if and only if all solutions of (B) are oscillatory.

Let us return to equation (A) of category II which is either strongly superlinear or strongly sublinear, and demonstrate that the Duality Principle makes it possible to find the desired oscillation criteria for (A) almost automatically from the already known oscillation criteria for (B) which is category I.

It is known that since (A) has the coefficients $\{p(t), q(t)\}$ and the exponent $\{\alpha(t), \beta(t)\}$, the components of the coefficients $\{\widetilde{p}(t), \widetilde{q}(t)\}$ and the exponents $\{\widetilde{\alpha}(t), \widetilde{\beta}(t)\}$ of (B) are expressed as

$$\widetilde{p}(t) = q(t)^{-\frac{1}{\beta(t)}}, \quad \widetilde{q}(t) = p(t)^{-\frac{1}{\alpha(t)}}, \quad \widetilde{\alpha}(t) = \frac{1}{\beta(t)}, \quad \widetilde{\beta}(t) = \frac{1}{\alpha(t)}.$$

Suppose that (A) is strongly superlinear. In addition suppose that $\beta(\infty) < \infty$. Then, (B) is also strongly superlinear and $\widetilde{\alpha}(\infty) = 1/\beta(\infty) > 0$, Since (B) is of category I, Theorem 2.2 is applicable to to (B) and ensures that all solutions of (B) are oscillatory if and only if

$$\int\limits_a^\infty \big(\widetilde{p}(t)^{-1}\widetilde{\rho}(t)\big)^{\frac{1}{\widetilde{\alpha}(t)}}\, dt = \infty.$$

Noting that

$$\widetilde{p}(t)^{-1} = q(t)^{\frac{1}{\alpha(t)}} \ \text{ and } \ \widetilde{\rho}(t) = \int\limits_t^\infty p(s)^{-\frac{1}{\alpha(s)}}\, ds = \pi_\alpha(t),$$

we are led to the following oscillation theorem for strongly superlinear equation (A) of category II.

Theorem 3.1. *Let* (A) *be a strongly superlinear equation with* $\beta(\infty) < \infty$ *and of category* II*. All of its solutions are oscillatory if and only if*

$$\int_a^\infty q(t)\pi_\alpha(t)^{\beta(t)}\,dt = \infty.$$

Next, suppose that (A) is strongly sublinear with $\beta(\infty) < \infty$. Then, (B) is also strongly sublinear with $\widetilde{\alpha}(\infty) > 0$ and so applying Theorem 2.3 to (B) we see that all solutions of (B) with $\widetilde{\alpha}(\infty) > 0$ are oscillatory if and only if

$$\int_a^\infty \widetilde{q}(t)\widetilde{P}_\alpha(t)^{\widetilde{\beta}(t)}\,dt = \infty. \tag{3.2}$$

Noting (3.2) that $\widetilde{q}(t) = p(t)^{-\frac{1}{\alpha(t)}}$ and

$$\widetilde{P}_{\widetilde{\alpha}(t)}(t)^{\widetilde{\beta(t)}} = \Big(\int_a^t q(s)\,ds\Big)^{\frac{1}{\alpha(t)}} = Q(t)^{\frac{1}{\alpha(t)}}.$$

we are led to the following oscillation theorem for strongly sublinear equation of category II.

Theorem 3.2. *Let* (A) *be a strongly sublinear equation with* $\beta(\infty) < \infty$ *and of category* II*. All of its solutions are oscillatory if and only if*

$$\int_a^\infty (p(t)^{-1}Q(t))^{\frac{1}{\alpha(t)}}\,dt = \infty.$$

Concluding Remarks. Recenly there has been an increasing interest in the study of differential equations with variable exponents. To the best of our knowledge the pioneer of oscillation theory of such equations is Koplatadze who published the papers [4, 5]. Koplatadze's results are closely related to ours specialized to equation (A) with $\alpha(t) \equiv 1$ and $p(t) \equiv 1$. For other related topics see e.g.the papers [2, 7, 8].

References

[1] Á. Elbert and T. Kusano, Oscillation and nonoscillation theorems for a class of second order quasilinear differential equations. *Acta Math. Hungar.* **56** (1990), no. 3-4, 325–336.

[2] K. Fujimoto and N. Yamaoka, Oscillation constants for Euler type differential equations involving the $p(t)$-Laplacian. *J. Math. Anal. Appl.* **470** (2019), no. 2, 1238–1250.

[3] J. Jaroš and T. Kusano, Nonoscillation theory of nonlinear differential equations of Emden–Fowler type with variable exponents (*submitted for publication*).

[4] R. Koplatadze, On oscillatory properties of solutions of generalized Emden–Fowler type differential equations. *Proc. A. Razmadze Math. Inst.* **145** (2007), 117–121.

[5] R. Koplatadze, Essentially nonlinear generalized differential equation of Emden–Fowler type with delay argument. *Semin. I. Vekua Inst. Appl. Math. Rep.* **35** (2009), 64–67.

[6] T. Kusano, A. Ogata and H. Usami, Oscillation theory for a class of second order quasilinear ordinary differential equations with application to partial differential equations. *Japan. J. Math. (N.S.)* **19** (1993), no. 1, 131–147.

[7] Y. Shoukaku, Oscillation criteria for nonlinear differential equations with $p(t)$-Laplacian. *Math. Bohem.* **141** (2016), no. 1, 71–81.

[8] N. Yoshida, Picone identities for half-linear elliptic operators with $p(x)$-Laplacians and applications to Sturmian comparison theory. *Nonlinear Anal.* **74** (2011), no. 16, 5631–5642.