Properties of Oscillatory Solutions of Second Order Half-Linear Differential Equations

Jaroslav Jaroš

Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University, Mlynská Dolina, Bratislava, Slovakia
E-mail: jaros@fmph.uniba.sk

Takaŝi Kusano

Department of Mathematics, Faculty of Science, Hiroshima University Higashi-Hiroshima, Japan
E-mail: kusanot@zj8.so-net.ne.jp

Tomoyuki Tanigawa

Department of Mathematics, Osaka Metropolitan University, Osaka, Japan
E-mail: Tanigawa.Tomoyuki@omu.ac.jp

1 Introduction

This report is a continuation of the note [1] that was submitted to the QUALITDE-2021 in which the distribution of zeros and extrema of oscillatory solutions were studied for second order half-linear differential equations of the form

$$\big(p(t)\varphi_\alpha(x')\big)' + q(t)\varphi_\alpha(x) = 0, \tag{HL}$$

where α is a positive constant, $p(t)$ and $q(t)$ are positive continuously differentiable functions on $[a,\infty)$, and φ_α stands for the odd function on $\mathbb{R}$ given by

$$\varphi_\alpha(u) = |u|^\alpha \operatorname{sgn} u = |u|^{\alpha-1}u, \ \ u \in \mathbb{R}.$$

We assume that equation (HL) is oscillatory, that is, all of its nontrivial solutions are oscillatory. Let $x(t)$ be a solution of (HL) on $[a,\infty)$. Let $\{\sigma_k\}_{k=1}^\infty$ ($\sigma_k < \sigma_{k+1}$) be the sequence of all zeros of $x(t)$, and let $\{\tau_k\}_{k=1}^\infty$ ($\tau_k < \tau_{k+1}$) be the sequence of all points at which $x(t)$ takes on its local extrema. It is clear that $x'(\tau_k) = 0$ for all k. The value $|x'(\sigma_k)|$ is called the *slope* of $x(t)$ at $t = \sigma_k$, while the value $|x(\tau_k)|$ is called the *amplitude* of $x(t)$ at $t = \tau_k$. The sets of the amplitudes and slopes of $x(t)$ determine the following

$$\mathcal{A}^*[x] = \sup_k |x(\tau_k)|, \quad \mathcal{A}_*[x] = \inf_k |x(\tau_k)|, \tag{1.1}$$

$$\mathcal{S}^*[x] = \sup_k |x'(\sigma_k)|, \quad \mathcal{S}_*[x] = \inf_k |x'(\sigma_k)|, \tag{1.2}$$

which provide helpful information about the oscillatory behavior of $x(t)$. Since it is difficult to analyze the equation (HL) with general positive functions $p(t)$ and $q(t)$, we restrict our analysis to the equation in which both $p(t)$ and $q(t)$ are positive monotone functions on $[a,\infty)$. The four possible cases

(i) $p'(t) \geqq 0$, $q'(t) \leqq 0$;

(ii) $p'(t) \leqq 0$, $q'(t) \geqq 0$;

(iii) $p'(t) \geqq 0$, $q'(t) \geqq 0$;

(iv) $p'(t) \leqq 0$, $q'(t) \leqq 0$

should be distinguished.

2 Known results

In this section we state known results for the convenience of the reader (see [1]).

Theorem A. *Let* (HL) *be oscillatory and let $x(t)$ be a solution of it satisfying the initial condition*

$$x(a) = l, \quad x'(a) = m, \tag{2.1}$$

where l and m are any given constants such that $(l, m) \neq (0, 0)$.

(i) *Suppose that $p'(t) \geqq 0$ and $q'(t) \leqq 0$ for $t \geqq a$. Then,*

$$\mathcal{A}^*[x] \leqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{q(\infty)}\Big]^{\frac{1}{\alpha+1}} \quad \text{if } q(\infty) > 0, \tag{2.2}$$

$$\mathcal{A}_*[x] \geqq \Big[\frac{p(a)^{\frac{1}{\alpha}}\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{p(\infty)^{\frac{1}{\alpha}} q(a)}\Big]^{\frac{1}{\alpha+1}} \quad \text{if } p(\infty) < \infty. \tag{2.3}$$

(ii) *Suppose that $p'(t) \leqq 0$ and $q'(t) \geqq 0$ for $t \geqq a$.* Then,

$$\mathcal{A}^*[x] \leqq \Big[\frac{p(a)^{\frac{1}{\alpha}}\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{p(\infty)^{\frac{1}{\alpha}} q(a)}\Big]^{\frac{1}{\alpha+1}} \quad \text{if } p(\infty) > 0, \tag{2.4}$$

$$\mathcal{A}_*[x] \geqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{q(\infty)}\Big]^{\frac{1}{\alpha+1}} \quad \text{if } q(\infty) < \infty. \tag{2.5}$$

(iii) *Suppose that $(p(t)^{\frac{1}{\alpha}} q(t))' \geqq 0$ for $t \geqq a$. Then,*

$$\mathcal{A}^*[x] \leqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{q(a)}\Big]^{\frac{1}{\alpha+1}}, \tag{2.6}$$

$$\mathcal{A}_*[x] \geqq \Big[\frac{p(a)^{\frac{1}{\alpha}}\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{p(\infty)^{\frac{1}{\alpha}} q(\infty)}\Big]^{\frac{1}{\alpha+1}} \quad \text{if } p(\infty)^{\frac{1}{\alpha}} q(\infty) < \infty. \tag{2.7}$$

(iv) *Suppose that $(p(t)^{\frac{1}{\alpha}} q(t))' \leqq 0$ for $t \geqq a$. Then,*

$$\mathcal{A}^*[x] \leqq \Big[\frac{p(a)^{\frac{1}{\alpha}}\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{p(\infty)^{\frac{1}{\alpha}} q(\infty)}\Big]^{\frac{1}{\alpha+1}} \quad \text{if } p(\infty)^{\frac{1}{\alpha}} q(\infty) > 0, \tag{2.8}$$

$$\mathcal{A}_*[x] \geqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{q(a)}\Big]^{\frac{1}{\alpha+1}}. \tag{2.9}$$

Since the constants l and m in (2.1) are arbitrary, the above inequalities (2.2)–(2.9) guarantee under the indicated conditions on $p(\infty)$ and/or $q(\infty)$ that $\mathcal{A}^*[x] < \infty$ and $\mathcal{A}_*[x] > 0$ for all solutions $x(t)$ of (HL). Then, noting that $\mathcal{A}^*[x] < \infty$ gives the boundedness of $x(t)$ on $[a, \infty)$ and more $\mathcal{A}^*[x] < \infty$ and $\mathcal{A}_*[x] > 0$ implies the non-decaying boundedness of $x(t)$ on $[a, \infty)$, we have the following propositions.

Corollary A. *Suppose that* (HL) *is oscillatory. All of its solutions are bounded on* $[a, \infty)$ *if* $p(t)$ *and* $q(t)$ *satisfy one of the following conditions*:

(i) $p'(t) \geqq 0$, $q'(t) \leqq 0$ *for* $t \geqq a$ *and* $q(\infty) > 0$;

(ii) $p'(t) \leqq 0$, $q'(t) \geqq 0$ *for* $t \geqq a$ *and* $p(\infty) > 0$;

(iii) $(p(t)^{\frac{1}{\alpha}} q(t))' \geqq 0$ *for* $t \geqq a$;

(iv) $(p(t)^{\frac{1}{\alpha}} q(t))' \leqq 0$ *for* $t \geqq a$ *and* $p(\infty)^{\frac{1}{\alpha}} q(\infty) > 0$.

Corollary B. *Suppose that* (HL) *is oscillatory. All of its solutions are non-decaying bounded on* $[a, \infty)$ *if* $p(t)$ *and* $q(t)$ *satisfy one of the following conditions*:

(i) $p'(t) \geqq 0$, $q'(t) \leqq 0$ *for* $t \geqq a$ *and* $p(\infty) < \infty$, $q(\infty) > 0$;

(ii) $p'(t) \leqq 0$, $q'(t) \geqq 0$ *for* $t \geqq a$ *and* $p(\infty) > 0$, $q(\infty) < \infty$;

(iii) $(p(t)^{\frac{1}{\alpha}} q(t))' \geqq 0$ *for* $t \geqq a$ *and* $p(\infty)^{\frac{1}{\alpha}} q(\infty) < \infty$;

(iv) $(p(t)^{\frac{1}{\alpha}} q(t))' \leqq 0$ *for* $t \geqq a$ *and* $p(\infty)^{\frac{1}{\alpha}} q(\infty) > 0$.

Theorem B. *Let* (HL) *be oscillatory and let* $x(t)$ *be a solution of it satisfying* (2.1).

(i) *Suppose that* $p'(t) \geqq 0$ *and* $q'(t) \leqq 0$ *for* $t \geqq a$. *Then,*

$$\mathcal{S}^*[x] \leqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{\alpha p(a)}\Big]^{\frac{1}{\alpha+1}}, \tag{2.10}$$

$$\mathcal{S}_*[x] \geqq \Big[\frac{p(a)^{\frac{1}{\alpha}} q(\infty)\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(\infty)^{1+\frac{1}{\alpha}} q(a)}\Big]^{\frac{1}{\alpha+1}} \quad \text{if } p(\infty) < \infty \text{ and } q(\infty) > 0. \tag{2.11}$$

(ii) *Suppose that* $p'(t) \leqq 0$ *and* $q'(t) \geqq 0$ *for* $t \geqq a$. *Then,*

$$\mathcal{S}^*[x] \leqq \Big[\frac{p(a)^{\frac{1}{\alpha}} q(\infty)\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(\infty)^{1+\frac{1}{\alpha}} q(a)}\Big]^{\frac{1}{\alpha+1}} \quad \text{if } p(\infty) > 0 \text{ and } q(\infty) < \infty, \tag{2.12}$$

$$\mathcal{S}_*[x] \geqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{\alpha p(a)}\Big]^{\frac{1}{\alpha+1}}. \tag{2.13}$$

(iii) *Suppose that* $p'(t) \geqq 0$ *and* $q'(t) \geqq 0$ *for* $t \geqq a$. *Then,*

$$\mathcal{S}^*[x] \leqq \Big[\frac{q(\infty)\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(a) q(a)}\Big]^{\frac{1}{\alpha+1}} \quad \text{if } q(\infty) < \infty, \tag{2.14}$$

$$\mathcal{S}_*[x] \geqq \Big[\frac{p(a)^{\frac{1}{\alpha}}\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(\infty)^{1+\frac{1}{\alpha}}}\Big]^{\frac{1}{\alpha+1}} \quad \text{if } p(\infty) < \infty. \tag{2.15}$$

(iv) *Suppose that $p'(t) \leqq 0$ and $q'(t) \leqq 0$ for $t \geqq a$. Then,*

$$\mathcal{S}^*[x] \leqq \Big[\frac{p(a)^{\frac{1}{\alpha}}\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(\infty)^{1+\frac{1}{\alpha}}}\Big]^{\frac{1}{\alpha+1}} \ \ \text{if } p(\infty) > 0, \tag{2.16}$$

$$\mathcal{S}_*[x] \geqq \Big[\frac{q(\infty)\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(a)q(a)}\Big]^{\frac{1}{\alpha+1}} \ \ \text{if } q(\infty) > 0. \tag{2.17}$$

Corollary C. *Let* (HL) *be oscillatory. If $p(t)$ and $q(t)$ are monotone functions such that $0 < p(\infty) < \infty$ and $0 < q(\infty) < \infty$, then $\mathcal{S}^*[x] < \infty$ and $\mathcal{S}_*[x] > 0$ for all solutions $x(t)$ of* (HL).

3 Main results

Our first result concerns the estimation of the derivatives of oscillatory solutions of (HL).

Theorem 3.1. *Let* (HL) *be oscillatory and let $x(t)$ be the solution of it satisfying the initial condition* (2.1).

(i) *$p'(t) \geqq 0$ and $q'(t) \leqq 0$ for $t \geqq a$. Then,*

$$\sup_t |x'(t)| \leqq \Big[\frac{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}}{\alpha p(a)}\Big]^{\frac{1}{\alpha+1}},$$

$$\lim_{t\to\infty} x'(t) = 0 \ \ \text{if } p(\infty) = \infty.$$

(ii) *$p'(t) \leqq 0$ and $q'(t) \geqq 0$ for $t \geqq a$. Then,*

$$\sup_t |x'(t)| \leqq \Big[\frac{p(a)^{\frac{1}{\alpha}} q(\infty)\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(\infty)^{1+\frac{1}{\alpha}} q(a)}\Big]^{\frac{1}{\alpha+1}} \ \ \text{if } p(\infty) > 0 \ \text{ and } \ q(\infty) < \infty.$$

(iii) *$p'(t) \geqq 0$ and $q'(t) \geqq 0$ for $t \geqq a$. Then,*

$$\sup_t |x'(t)| \leqq \Big[\frac{q(\infty)\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{p(a)q(a)}\Big]^{\frac{1}{\alpha+1}} \ \ \text{if } q(\infty) < \infty,$$

$$\lim_{t\to\infty} x'(t) = 0 \ \ \text{if } \lim_{t\to\infty} \frac{q(t)}{p(t)} = 0.$$

(iv) *$p'(t) \leqq 0$ and $q'(t) \leqq 0$ for $t \geqq a$. Then,*

$$\sup_t |x'(t)| \leqq \Big[\frac{p(a)^{\frac{1}{\alpha}}\{q(a)|l|^{\alpha+1} + \alpha p(a)|m|^{\alpha+1}\}}{\alpha p(\infty)^{1+\frac{1}{\alpha}}}\Big]^{\frac{1}{\alpha+1}} \ \ \text{if } p(\infty) > 0.$$

Our next result in this section concerns the sequences of zeros of solutions of (HL). We are interested in explicit laws or rules, if any, governing the arrangement of this sequences. Assume that (HL) is oscillatory. Let $x(t)$ be any of its solutions on $[a, \infty)$ and let $\{\sigma_k\}$ represent the sequences of zeros of $x(t)$.

Theorem 3.2. *The sequence $\{\sigma_{k+1} - \sigma_k\}$ is decreasing or increasing according to $p'(t) \leqq 0$ and $q'(t) \geqq 0$, or $p'(t) \geqq 0$ and $q'(t) \leqq 0$ for $t \geqq a$.*

4 Example

Consider the half-linear differential equation

$$\big((\coth(t+\tau))^{\alpha}\varphi_{\alpha}(x')\big)' + k\tanh(t+\tau)\varphi_{\alpha}(x) = 0 \tag{4.1}$$

on $[0,\infty)$, where $\tau \geqq 0$ and $k > 0$ are constants. Equation (4.1) is oscillatory since the functions $p(t) = (\coth(t+\tau))^{\alpha}$ and $q(t) = k\tanh(t+\tau)$ are not integrable on $[0,\infty)$. It is clear that $p(t)$ and $q(t)$ satisfy $p'(t) \leqq 0$, $q'(t) \geqq 0$, $(p(t)^{\frac{1}{\alpha}}q(t))' = 0$, $p(0) = (\coth\tau)^{\alpha}$, $p(\infty) = 1$, $q(0) = k\tanh\tau$ and $q(\infty) = k$, all nontrivial solutions of equation (4.1) are bounded and non-decaying by (ii) and (iii) of Corollary A and Corollary B, respectively. As regards the estimates for upper and lower amplitudes and upper and lower slopes of solutions of (4.1), we obtain, for example,

$$\begin{aligned}
\mathcal{A}^*[x] &\leqq \Big[\coth\tau\,|l|^{\alpha+1} + \frac{\alpha}{k}(\coth\tau)^{\alpha+2}|m|^{\alpha+1}\Big]^{\frac{1}{\alpha+1}},\\
\mathcal{A}_*[x] &\geqq \Big[\tanh\tau\,|l|^{\alpha+1} + \frac{\alpha}{k}(\coth\tau)^{\alpha}|m|^{\alpha+1}\Big]^{\frac{1}{\alpha+1}}
\end{aligned}$$

from (ii) of Theorem A, and

$$\begin{aligned}
\mathcal{S}^*[x] &\leqq \Big[\frac{k}{\alpha}\coth\tau\,|l|^{\alpha+1} + (\coth\tau)^{\alpha+2}|m|^{\alpha+1}\Big]^{\frac{1}{\alpha+1}},\\
\mathcal{S}_*[x] &\geqq \Big[\frac{k}{\alpha}(\tanh\tau)^{\alpha+1}|l|^{\alpha+1} + |m|^{\alpha+1}\Big]^{\frac{1}{\alpha+1}}
\end{aligned}$$

from (ii) of Theorem B. If in particular $\tau \to \infty$ and $k = \alpha$, then the upper and lower amplitudes and slopes coincide, that is,

$$\mathcal{A}^*[x] = \mathcal{A}_*[x] = \mathcal{S}^*[x] = \mathcal{S}_*[x] = \big[\,|l|^{\alpha+1} + |m|^{\alpha+1}\big]^{\frac{1}{\alpha+1}}.$$

This value may well be called the amplitude $\mathcal{A}[x]$ and the slope $\mathcal{S}[x]$ of the solution $x(t)$ of the equation

$$(\varphi_{\alpha}(x'))' + \alpha\varphi_{\alpha}(x) = 0. \tag{4.2}$$

Equation (4.2) is known as a differential equation generating a generalized trigonometric function. Its solution $x(t)$ determined by the initial condition $x(0) = 0$, $x'(0) = 1$ is the generalized sine function $x(t) = S(t)$ which exists on $\mathbb{R}$, is periodic with period $2\pi_{\alpha}$, $\pi_{\alpha} = \frac{2\pi}{\alpha+1}/\sin(\frac{\pi}{\alpha+1})$, and vanishes at $t = n\pi_{\alpha}$, $n \in \mathbb{Z}$, whose amplitude and slope are given by $\mathcal{A}[x] = 1$ and $\mathcal{S}[x] = 1$, respectively. Moreover, from the first statement of Theorem 3.2 applied to equation (4.2), it follows that the sequences $\{\sigma_k\}$ of zeros of any solution of it are arranged in such a way that $\{\sigma_{k+1} - \sigma_k\}$ is decreasing.

References

[1] J. Jaroš, T. Kusano and T. Tanigawa, Oscillatory properties of solutions of second order half-linear differential equations. *Abstracts of the International Workshop on the Qualitative Theory of Differential Equations – QUALITDE-2021*, Tbilisi, Georgia, December 18-20, pp. 83–86; `https://rmi.tsu.ge/eng/QUALITDE-2021/Jaros_Kusano_Tanigawa_workshop_2021.pdf`.