On One Diffusion System of Nonlinear Partial Differential Equations

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The present note is devoted to one one-dimensional system of nonlinear partial differential equations (SNPDE). Many mathematical models in physics, biology, engineering and so on are described by such type of models (see, for example, [1–3, 6, 9, 11, 12, 15, 17, 18] and the references therein). In this article the initial-boundary value problems are considered and some features of solutions are stated. The finite difference scheme is constructed for the investigated problem and the question of its convergence is given. A lots of scientific works are dedicated to the investigation and numerical resolution of such models (see, for example, [1–13, 15, 17, 18] and the references therein).

As a model, let us consider the SNPDE of the following type:

$$\begin{aligned} &\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(A(V)\,\frac{\partial U}{\partial x}\Big), \\ &\frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\Big(C(V)\,\frac{\partial V}{\partial x}\Big) + F\Big(V, \frac{\partial U}{\partial x}\Big), \end{aligned} \tag{1}$$

where A, C and F are the given functions of their arguments.

The numerous diffusion problems are reduced to (1) SNPDE. In particular, if

$$C(V) \equiv 0, \quad F\Big(V, \frac{\partial U}{\partial x}\Big) = B(V)\Big(\frac{\partial U}{\partial x}\Big)^2,$$

(1) SNPSE meets at the modeling of penetration of an electromagnetic field into a medium, whose coefficient of electroconductivity depends on temperature, without taking into account the heat conductivity [11]. If

$$C(V) \not\equiv 0, \quad F\Big(V, \frac{\partial U}{\partial x}\Big) = -D(V) + B(V)\Big(\frac{\partial U}{\partial x}\Big)^2,$$

where B and D are given functions of their arguments, then system (1) describes the process of penetration of an electromagnetic field into the medium, taking into account the heat conductivity [11].

If

$$A(V) \equiv V, \quad C(V) \equiv 0, \quad F\Big(V, \frac{\partial U}{\partial x}\Big) = -V + G\Big(V\,\frac{\partial U}{\partial x}\Big),$$

where $0 < g_0 \le G(\xi) \le G_0, g_0$ and G_0 are constants, and G is a smooth enough function, then (1) represents an one-dimensional analogue of system which arises in studying the process of vein formation in young leaves of higher plants [15].

Let us consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(V^{\alpha}\,\frac{\partial U}{\partial x}\Big), \tag{2}$$

$$\frac{\partial V}{\partial t} = V^\alpha \Big(\frac{\partial U}{\partial x}\Big)^2, \tag{3}$$

$$U(0,t) = 0, \quad U(1,t) = \psi, \tag{4}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \tag{5}$$

where $\psi = const > 0$, and $U_0 = U_0(x), V_0 = V_0(x)$ are the given functions.

If $U(x,0) = \psi x$ and $V(x,0) = \delta_0 = const > 0$, as it is mentioned in [7] the pair of functions:

$$U(x,t) = \psi x, \quad V(x,t) = \big[\delta_0^{1-\alpha} + (1-\alpha)\psi^2 t\big]^{\frac{1}{1-\alpha}}, \tag{6}$$

is the solution of the initial-boundary value problem (2)–(5) for any $\alpha \neq 1$. However, if $\alpha > 1$, then for a finite time $t_0 = \delta_0^{1-\alpha}/(\psi^2(\alpha-1))$, the function V becomes unbounded. This example shows that the solutions of a system such as (2), (3) with smooth initial and boundary conditions can blow-up at a finite time.

Note that the functions U and V, determined by formulas (6), also satisfy the system:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(V^\alpha \frac{\partial U}{\partial x}\Big), \tag{7}$$

$$\frac{\partial V}{\partial t} = V^\alpha \Big(\frac{\partial U}{\partial x}\Big)^2 + \frac{\partial^2 V}{\partial x^2}, \tag{8}$$

with the boundary and initial conditions (4), (5) and adding to them the following boundary conditions:

$$\frac{\partial V}{\partial x}\Big|_{x=0} = \frac{\partial V}{\partial x}\Big|_{x=1} = 0. \tag{9}$$

From this we can conclude that if $\alpha > 1$, then for problem (4), (5), (7)–(9), the theorem on the existence of the global solution also does not hold.

The question of the stability of the stationary solution for appropriate diffusion problems is interesting for a mathematical explanation. In this connection we consider the following initial-boundary value problem:

$$\begin{gathered} \frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\Big(V^\alpha \frac{\partial U}{\partial x}\Big), \\ \frac{\partial V}{\partial t} = -V^\beta + V^\gamma \Big(\frac{\partial U}{\partial x}\Big)^2, \\ U(0,t) = 0, \quad V^\alpha \frac{\partial U}{\partial x}\Big|_{x=1} = \psi, \\ U(x,0) = U_0(x), \quad V(x,0) = V_0(x). \end{gathered} \tag{10}$$

It is easy to be convinced that the stationary solution of problem (10) has the form

$$\big(\psi^{\frac{\beta-\gamma}{2\alpha+\beta-\gamma}} x, \psi^{\frac{2}{2\alpha+\beta-\gamma}}\big).$$

The following statement is true.

Theorem 1. *If $\alpha \neq 0, 2\alpha+\beta-\gamma > 0$, then the stationary solution $\big(\psi^{\frac{\beta-\gamma}{2\alpha+\beta-\gamma}} x, \psi^{\frac{2}{2\alpha+\beta-\gamma}}\big)$ of problem* (10) *is linearly stable if and only if the following condition takes place*

$$(\gamma-\beta)\psi^{\frac{2(\beta-\alpha-1)}{2\alpha+\beta-\gamma}} < \frac{\pi}{4}. \tag{11}$$

Remark 1. If $\gamma - \beta \le 0$, then the stationary solution of problem (10) is always linearly stable.

Remark 2. Let
$$\gamma-\beta>0,\quad \beta-\alpha-1\neq 0,\quad \psi_c=\Big[\frac{\pi^2}{4(\gamma-\beta)}\Big]^{\frac{2\alpha+\beta-\gamma}{2(\beta-\alpha-1)}}.$$
Applying (11), if $0<\psi<\psi_c$, the stationary solution of problem (10) is linearly stable, and if $\psi>\psi_c$, it becomes unstable. For $\psi=\psi_c$, there is the possibility of occurrence of the Hopf-type bifurcation [14]. Small perturbations of the stationary solution can be transformed into a periodic in time self-oscillation.

Based on [5], the analogous investigations for more general models are given in [6, 8, 10].

Let us now consider the global stability of a solution of problem (10) for one particular case. Consider the following problem:
$$\begin{gathered}\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\Big(V\frac{\partial U}{\partial x}\Big),\\ \frac{\partial V}{\partial t}=-V+\Big(\frac{\partial U}{\partial x}\Big)^2,\\ U(0,t)=0,\quad V\,\frac{\partial U}{\partial x}\Big|_{x=1}=\psi,\\ U(x,0)=U_0(x),\quad V(x,0)=V_0(x).\end{gathered}\tag{12}$$

It is obvious that the stationary solution of problem (12) looks like to $(\psi^{1/3}x,\psi^{2/3})$. Introduce the notations:
$$y(x,t)=U(x,t)-\psi^{1/3}x,\quad z(x,t)=V(x,t)-\psi^{2/3},$$
where (U,V) is a solution of problem (12). We finally arrive at
$$\int\limits_0^1\big[y^2(x,t)+z^2(x,t)\big]\,dx\le e^{-Kt}\int\limits_0^1\Big\{\big[U_0(x)-\psi^{1/3}x\big]^2+\big[V_0(x)-\psi^{2/3}\big]^2\Big\}\,dx,$$
where K is a positive constant.

Thus, the following statement is true.

Theorem 2. *For the stationary solution of problem* (12) $(\psi^{1/3}x,\psi^{2/3})$ *there takes place the global and monotone stability in* $L_2(0,1)$.

Note that it is not difficult to get a certain generalization of the results considered here for the diffusion model, where the process of heat conductivity is taken into account.

Note also that some results regarding solvability, uniqueness, asymptotic behavior of solutions and properties of the difference schemes of corresponding integro-differential models with different kinds of boundary conditions for the above-mentioned equations and systems are studied in many works (see, for example, [6, 9] and the references therein).

Now, let us consider the convergence of difference schemes for the following problem:
$$\begin{gathered}\frac{\partial U}{\partial t}=\frac{\partial}{\partial x}\Big(V^\alpha\,\frac{\partial U}{\partial x}\Big),\\ \frac{\partial V}{\partial t}=V^{\alpha-1}\Big(\frac{\partial U}{\partial x}\Big)^2,\end{gathered}\tag{13}$$
$$\begin{gathered}U(0,t)=U(1,t)=0,\quad U(x,0)=U_0(x),\\ V(x,0)=V_0(x)\ge\sigma_0=const>0.\end{gathered}\tag{14}$$

The grid-function $u=\{u_i\}$ corresponding to U is considered in usual grid, whereas the function $v=\{v_i\}$ approximating V is considered at the centers of grid points.

Using usual notations [16], let us consider the following two-parameterized finite difference scheme:

$$
\begin{gathered}
u_t + \beta\tau u_{\bar{t}t} = \big[(v^{(\sigma)})^{\alpha} u_{\bar{x}}^{(\sigma)}\big]_x, \\
v_t + \beta\tau v_{\bar{t}t} = (v^{(\sigma)})^{\alpha-1} (u_{\bar{x}}^{(\sigma)})^2, \\
u(0,t) = u(1,t) = 0, \quad u(x,0) = U_0(x), \\
u(x,\tau) = U_0(x) + \tau\big[(V^{(\sigma)})^{\alpha} U_{\bar{x}}^{(\sigma)}\big]_{x,t=0}, \\
v(x,0) = V_0(x), \quad v(x,\tau) = V_0(x) + \tau\big[(V^{(\sigma)})^{\alpha-1} (U_{\bar{x}}^{(\sigma)})^2\big]_{t=0}.
\end{gathered}
\tag{15}
$$

Here,

$$
v^{(\sigma)} = \sigma v^{j+1} + (1-\sigma) v^j.
$$

The scheme (15), for the sufficiently smooth solution of the problem (13), (14), has the following order of approximation:

$$
O(\tau^2 + h^2 + (\sigma - 0,5 - \beta)\tau).
$$

Using the method of energy inequalities [16] for investigation of the difference schemes, the following statement is proved.

Theorem 3. *If problem* (13), (14) *has the sufficiently smooth solution, then the solution of the difference scheme* (15) *tends to the solution of problem* (13), (14) *and the rate of the convergence is* $O(\tau^2 + h^2 + (\sigma - 0,5 - \beta)\tau)$.

Remark 3. If $\sigma = 1/2, \beta = 0$, the two-layer difference scheme with accuracy of order $O(\tau^2 + h^2)$ is constructed. The same accuracy takes place if $\sigma = 1$ and $\beta = 1/2$. In this case, (15) is the three-layer scheme.

The difference scheme (15) is the system of the nonlinear algebraic equations. To be convinced of the solvability, it is enough to use an a-priori estimation which follows after the multiplication of equations (15) by u and v, respectively, and apply the Brouwer fixed-point lemma (see, e.g., [13]). Note that applying the same technique as we use in proving the convergence Theorem 3, it is not difficult to prove the uniqueness of the solution and the stability of the scheme (15).

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