Anti-Perron Effect of Changing Characteristic Exponents in Differential Systems

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The anti-Perron effect [1–3] (opposite to the well-known Perron one [4, 5]) presupposed the change of all positive characteristic exponents $\lambda_1(A) \leq \cdots \leq \lambda_n(A)$ of linear approximation

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \geq t_0, \tag{1}$$

with the bounded infinitely differentiable coefficients to negative in (some) nontrivial solutions of the differential system

$$\dot{y} = A(t)y + f(t,y), \quad y \in \mathbb{R}^n, \quad t \geq t_0, \tag{2}$$

also with an infinitely differentiable vector-function from the known classes of small perturbations. This effect is of great interest in its applications as compared with the Peron effect (devoted in a cycle of author's works). In the present report, we give an account of the results obtained by the author for the realization of anti-Prron effect.

1^0. In a class of linear exponentially decreasing perturbations the following theorem is valid.

Theorem 1 ([1]). *For any parameters $\lambda_n \geq \cdots \geq \lambda_1 > 0$, $\theta > 1$, $0 < \sigma < \lambda_1 + \theta^{-1}\lambda_2$, there exist:*

1) *system* (1) *with exponents $\lambda_i(A) = \lambda_i$, $i = \overline{1,n}$;*

2) *a linear perturbation $f(t,y) \equiv Q(t)y$ with the exponent $\lambda[Q] \leq -\sigma < 0$ such that system* (2) *has exactly $n-1$ linearly independent solutions $Y_1(t), \dots, Y_{n-1}(t)$ with the Liapounov exponents*

$$\lambda[Y_i] = \big[\theta(\sigma - \lambda_1) - \lambda_{i+1}\big](\theta - 1)^{-1}, \quad i = \overline{1, n-1}.$$

Remark 1. The variant $\lambda_1(A) > 0$, $\lambda_n(A+Q) < 0$, $\lambda[Q] < 0$ remains open.

2^0. In the case of linear perturbations $Q(t) \to 0$, as $t \to +\infty$, the following theorem is valid.

Theorem 2 ([2]). *For any parameters $0 < \lambda_1 \leq \cdots \leq \lambda_n$, $\mu_1 \leq \cdots \leq \mu_n < 0$, there exist:*

1) *system* (1) *with the exponents $\lambda_i(A) = \lambda_i$, $i = \overline{1,n}$;*

2) *the perturbation $Q(t) \to 0$, $t \to +\infty$ such that $\lambda_i(A+Q) = \mu_i$, $i = \overline{1,n}$.*

3^0. In the case of nonlinear m-perturbations

$$\|f(t,y)\| \le C_f\|y\|^m, \;\; m>1, \;\; y\in\mathbb{R}^n, \;\; t\ge t_0, \tag{3}$$

the following theorem holds.

Theorem 3 ([3])**.** *For any parameters $m>1$, $\theta>1$ and $\lambda>0$, there exist:*

1) *two-dimensional system* (1) *with exponents $\lambda_1(A)=\lambda_2(A)=\lambda>0$;*

2) *an infinitely differentiable perturbation* (3) *such that the nonlinear system* (2) *has the solution $Y(t)$ with the exponent*
$$\lambda[Y] = -\frac{\lambda(\theta+1)}{m\theta-1}\,.$$

The anti-Perron effect in the case under consideration is realized for a great number of solutions of the perturbed system. These systems belong to the spatially-time octants

$$R_1^2=\big\{y\in\mathbb{R}^2: \;\; y_1\ge 0, \;\; y_2\ge 0\big\}\times T_0, \quad R_2^2=\big\{y\in\mathbb{R}^2: \;\; y_1\le 0, \;\; y_2\ge 0\big\}\times T_0,$$
$$R_3^2=\big\{y\in\mathbb{R}^2: \;\; y_1\le 0, \;\; y_2\le 0\big\}\times T_0, \quad R_4^2=\big\{y\in\mathbb{R}^2: \;\; y_1\ge 0, \;\; y_2\le 0\big\}\times T_0,$$

in which $y=(y_1,y_2)\in\mathbb{R}^2$, $T_0=[t_0,+\infty)$, $t_0\ge 0$.

The following theorem is valid.

Theorem 4. *For any parameters $\lambda>0$, $m_4\ge m_3\ge m_2\ge m_1>1$, $\theta>1$, there exist:*

1) *two-dimensional linear system* (1) *with the characteristic exponents $\lambda_1(A)=\lambda_2(A)=\lambda>0$;*

2) *an infinitely differentiable m_1-perturbation $f(t,y):[t_0,+\infty)\times\mathbb{R}^2\to\mathbb{R}^2$ which is simultaneously an m_i-perturbation in the octant R_i^2 for any $i=\overline{1,4}$ such that the perturbed system* (2) *has the solutions $Y_i\subset R_i^2$, $i=\overline{1,4}$, with exponents*
$$\lambda[Y_i] = -\lambda\,\frac{\theta+1}{m_i\theta-1}<0.$$

Remark 2. An analogous to Theorem 3 statement on the existence of two-dimensional systems (1) with all positive exponents and (2) with perturbation (3) having 4 nontrivial solutions with negative different Liapounov exponents, is valid.

References

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