Bounded Solutions of a Linear Differential Equation with Piecewise Constant Operator Coefficients

Mykhailo Horodnii

Taras Shevchenko National University of Kyiv, Kyiv, Ukraine
E-mail: horodnii@gmail.com

Abstract

By passing to the corresponding difference equation, a criterion is obtained for the existence of a unique solution bounded on the entire real axis of a linear differential equation with piecewise constant operator coefficients.

1 Introduction

Let $(X, \|\cdot\|)$ be a complex separable Banach space, $L(X)$ be the Banach space of linear continuous operators acting from X into X, I and O be the identity and null operators in X, and $C_b(\mathbb{R}, X)$ be the Banach space of functions $x : \mathbb{R} \to X$ continuous and bounded on $\mathbb{R}$ with the norm

$$\|x\|_\infty = \sup_{t\in\mathbb{R}} \|x(t)\|.$$

Let us fix a natural number p, operators A, B; A_n, $1 \le n \le p$, from $L(X)$, real numbers $t_0 < t_1 < \dots < t_p$ and consider the differential equation

$$\begin{cases} x'(t) = Ax(t) + y(t), & t \ge t_p, \\ x'(t) = A_n x(t) + y(t), & t_{n-1} \le t < t_n, \ \ 1 \le n \le p, \\ x'(t) = Bx(t) + y(t), & t < t_0, \end{cases} \tag{1.1}$$

in which y is a fixed function from $C_b(\mathbb{R}, X)$. A bounded solution of equation (1.1) is a function $x \in C_b(\mathbb{R}, X)$ such that for each $t \in \mathbb{R} \setminus \{t_0, t_1, \dots, t_p\}$ there exists $x'(t)$ and equality (1.1) holds.

Our goal is to obtain necessary and sufficient conditions on the operator coefficients A, B; A_n, $1 \le n \le p$, for the following condition to be satisfied.

Boundedness condition. For each function $y \in C_b(\mathbb{R}, X)$ the differential equation (1.1) has a unique bounded solution.

2 Auxiliary statements

Consider the corresponding to (1.1) difference equation

$$\begin{cases} u_{n+1} = e^{A} u_n + v_n, & n \ge p, \\ u_{n+1} = e^{A_{n+1}(t_{n+1}-t_n)} u_n + v_n, & 0 \le n \le p-1, \\ u_{n+1} = e^{B} u_n + v_n, & n \le -1, \end{cases} \tag{2.1}$$

in which $\{v_n, n \in \mathbb{Z}\}$ is a given and $\{u_n, n \in \mathbb{Z}\}$ is a sought sequence of elements of the space X. We will say that the difference equation (2.1) satisfies the boundedness condition if it has a unique bounded solution $\{u_n, n \in \mathbb{Z}\}$ for each bounded sequence $\{v_n, n \in \mathbb{Z}\}$.

The following theorem holds.

Theorem 2.1. *For the differential equation* (1.1) *to satisfy the boundedness condition, it is necessary and sufficient that difference equation* (2.1) *also satisfy the boundedness condition.*

Let $S=\{z\in\mathbb{C} \mid \ |z|=1\}$. Let T be an operator from $L(X)$ such that $\sigma(T)\cap S=\varnothing$; $\sigma_-(T)$, $\sigma_+(T)$ are the parts of the spectrum of T that lie inside and outside the circle S, respectively; $P_-(T)$ and $P_+(T)$ are the Riesz projectors corresponding to $\sigma_-(T)$ and $\sigma_+(T)$. Then the space X can be decomposed into a direct sum $X=X_-(T)\dot{+}X_+(T)$ of subspaces $X_\pm(T)=P_\pm(T)(X)$ that are invariant with respect to T (see, for example, [3, pp. 32–34]).

For brevity, we denote $E_k=e^{A_k(t_k-t_{k-1})}$, $E_{jk}=E_kE_{k-1}\dots E_j$, $1\le j\le k\le p$. Since the operator E_{jk} is continuously invertible, the image $E_{jk}(G)$ of an arbitrary subspace G of a Banach space X is also a subspace. Therefore, by Theorem 3 of [4], the following theorem holds.

Theorem 2.2. *For the difference equation* (2.1)*, the boundedness condition is satisfied if and only if the following conditions are satisfied:*

(i1) $\sigma(e^A)\cap S=\varnothing$, $\sigma(e^B)\cap S=\varnothing$;

(i2) $X=X_-(e^A)\dot{+}E_{1p}(X_+(e^B))$.

3 Main results

Now let $i\mathbb{R}=\{it \mid \ t\in\mathbb{R}\}$, $V\in L(X)$, $\sigma(V)\cap i\mathbb{R}=\varnothing$, $\widetilde{\sigma}_-(V)$, $\widetilde{\sigma}_+(V)$ be the parts of the spectrum of the operator V that lie in the left and right half-planes of $\mathbb{C}$, respectively. Then, as for the operator T, the space X decomposes into a direct sum $X=\widetilde{X}_-(V)\dot{+}\widetilde{X}_+(V)$ of subspaces $\widetilde{X}_\pm(V)=\widetilde{P}_\pm(T)(V)$ that are invariant with respect to the operator V, where $\widetilde{P}_\pm(V)$ are the Riesz projections corresponding to $\widetilde{\sigma}_\pm(V)$. The above statements allow us to prove the following theorems.

Theorem 3.1. *For the differential equation* (1.1) *to satisfy the boundedness condition, it is necessary and sufficient that the following conditions be satisfied:*

(j1) $\sigma(A)\cap i\mathbb{R}=\varnothing$, $\sigma(B)\cap i\mathbb{R}=\varnothing$;

(j2) $X=\widetilde{X}_-(A)\dot{+}E_{1p}(\widetilde{X}_+(B))$.

Theorem 3.2. *Assume that conditions j*1), *j*2) *of Theorem 3 are satisfied. Then the following statements hold:*

(b1) *for each* $0\le k\le p$,
$$X=E^{-1}_{(k+1)p}(\widetilde{X}_-(A))\dot{+}E_{1k}(\widetilde{X}_+(B)), \tag{3.1}$$
where $E_{(p+1)p}=E_{10}=I$;

(b2) *corresponding to the function* $y\in C_b(\mathbb{R},X)$ *the unique bounded solution* x *of the differential equation* (1.1) *has the following form:*

if $t\ge t_p$, *then*

$$x(t)=\int\limits_{t_p}^{t}e^{A(t-s)}P_-(A)y(s)\,ds-\int\limits_{t}^{+\infty}e^{A(t-s)}P_+(A)y(s)\,ds$$
$$+\,e^{A(t-t_p)}P_p^-\bigg(\int\limits_{t_p}^{+\infty}e^{A(t_p-s)}P_+(A)y(s)\,ds+\sum_{k=1}^{p}E_{kp}\int\limits_{t_{k-1}}^{t_k}e^{A_k(t_{k-1}-s)}y(s)\,ds$$

$$+ E_{1p}P_0^- \int_{-\infty}^{t_0} e^{B(t_0-s)} P_-(B)y(s)\,ds\Big);$$

if $1 \le k \le p$, then (sequentially in descending order from $k = p$ to $k = 1$)

$$x(t) = -\int_{t}^{t_k} e^{A_k(t-s)} y(s)\,ds + e^{A_k(t-t_k)} x(t_k), \quad t \in [t_{k-1}; t_k);$$

if $t < t_0$, then

$$x(t) = \int_{-\infty}^{t} e^{B(t-s)} P_-(B)y(s)\,ds - \int_{t}^{t_0} e^{B(t-s)} P_+(B)y(s)\,ds$$
$$- e^{B(t-t_0)}\Big(E_{1p}^{-1}P_p^+ \int_{t_p}^{+\infty} e^{A(t_p-s)} P_+(A)y(s)\,ds$$
$$+ \sum_{k=1}^{p} E_{1(k-1)}^{-1} P_{k-1}^+ \int_{t_{k-1}}^{t_k} e^{A_k(t_{k-1}-s)} y(s)\,ds + P_0^+ \int_{-\infty}^{t_0} e^{B(t_0-s)} P_-(B)y(s)\,ds\Big).$$

Here for each $0 \le k \le p$ P_k^-, P_k^+ are the projectors corresponding to representation (3.1).

For a differential equation with a variable operator coefficient, the boundedness condition was studied, in particular, in [1–3,5] using the exponential dichotomy condition on $\mathbb{R}$ for the corresponding homogeneous differential equation. In the general case, checking the exponential dichotomy condition is not trivial. Theorem 3.1 contains necessary and sufficient conditions directly on the operator coefficients that ensure that the exponential dichotomy condition is satisfied for the homogeneous differential equation corresponding to equation (1.1).

Acknowledgement

The reported study was partially supported by Ministry of Education and Science of Ukraine: Grant of the Ministry of Education and Science of Ukraine for perspective development of a scientific direction *"Mathematical Sciences and Natural Sciences"* at Taras Shevchenko National University of Kyiv.

References

[1] A. G. Baskakov, Investigation of linear differential equations by the methods of the spectral theory of difference operators and linear relations. (Russian) *Uspekhi Mat. Nauk* **68** (2013), no. 1(409), 77–128; translation in *Russian Math. Surveys* **68** (2013), no. 1, 69–116.

[2] W. A. Coppel, Dichotomies and reducibility. *J. Differential Equations* **3** (1967), 500–521.

[3] Ju. L. Dalec'kiĭ and M. G. Kreĭn, *Stability of Solutions of Differential Equations in Banach space*. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 43. American Mathematical Society, Providence, RI, 1974.

[4] M. Horodnii, Bounded solutions of a difference equation with piecewise constant operator coefficients. (Ukrainian) *Nelīnīĭnī Koliv.* **26** (2023), no. 2, 199–209; translation in *J. Math. Sci. (N.Y.)* **279** (2024), no. 3, 330–342.

[5] J. L. Massera and J. J. Schäffer, *Linear Differential Equations and Function Spaces.* Pure and Applied Mathematics, Vol. 21. Academic Press, New York–London, 1966.