The Inverse Problem for Periodic Travelling Waves of the Linear 1D Shallow-Water Equations

Robert Hakl

Institute of Mathematics, Czech Academy of Sciences
Brno, Czech Republic
E-mail: hakl@ipm.cz

Pedro J. Torres

Departamento de Matemática Aplicada & Modeling Nature (MNat) Research Unit.,
Universidad de Granada, Granada, Spain
E-mail: ptorres@ugr.es

1 Introduction

The motion of small amplitude waves of a water layer with variable depth along the x-axis is described by the equations of the shallow water theory

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}[h(x)u] = 0, \quad \frac{\partial u}{\partial t} + g\,\frac{\partial \eta}{\partial x} = 0,$$

where $\eta(x,t)$ is the vertical water surface elevation, $u(x,t)$ is the depth-averaged water flow velocity (also called wave velocity), $h(x)$ is the unperturbed water depth and g is the gravity acceleration (see Fig. 1). From now on, we assume without loss of generality that $g = 1$.

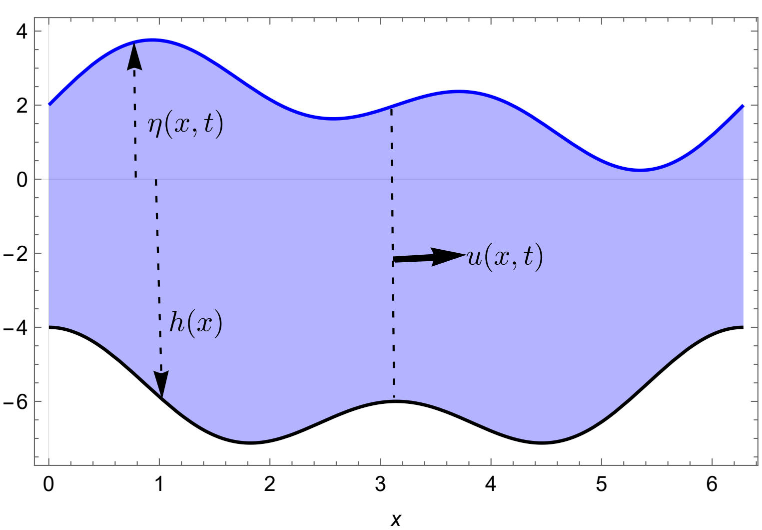


Figure 1. Graphical description of the model.

The shallow water equations constitute a system of coupled PDEs of first order that can be easily decoupled into a single wave equation for the surface displacement

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x}\left[h(x)\,\frac{\partial \eta}{\partial x}\right] = 0, \tag{1.1}$$

or for the water velocity

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2}{\partial x^2}\,[h(x)u] = 0. \tag{1.2}$$

There is a considerable number of papers devoted to finding sufficient conditions on the bottom profile $h(x)$ to ensure the existence of travelling waves or other explicit solutions [1–7, 9, 10]. A travelling wave is a special solution of the form $q(x)\exp i[\omega t - \Psi(x)]$, where both q and Ψ are scalar real-valued functions. In the related literature, $q(x)$ is known as the amplitude of the travelling wave, ω is the frequency and $\Psi(x)$ is the phase, which is called non-trivial if it is non-constant. In this paper, we are going to study the following inverse problem: *given a prescribed amplitude $q(x)$, can we determine a suitable bottom profile $h(x)$ allowing the equation to admit a travelling wave with amplitude $q(x)$?*

2 The inverse problem for water velocity

From now on, C_T^+ will denote the space of continuous scalar T-periodic functions with positive values. In this section, we study the inverse problem for the water velocity. Given a fixed $q \in C_T^+$, we wonder if there exists $h \in C_T^+$ such that Eq. (1.2) has a travelling wave

$$u(x,t) = q(x)\exp i\big[\omega t - \Psi(x)\big]. \tag{2.1}$$

Inserting (2.1) into (1.2) and separating real and imaginary parts, we get the equations

$$(hq)'' + \omega^2 q - hq\Psi'^2 = 0, \tag{2.2}$$

$$2(hq)'\Psi' + hq\Psi'' = 0. \tag{2.3}$$

From (2.3), we deduce that $[(hq)^2\Psi']' = 0$, and $(hq)^2\Psi'$ is a conserved quantity, which is actually an energy flux, and it is in total analogy with the angular momentum in systems with radial symmetry. This means that there exists $\alpha \in \mathbb{R}$ such that

$$\big[h(x)q(x)\big]^2\Psi'(x) = \alpha, \ \ \forall\, x \in \mathbb{R}. \tag{2.4}$$

Now, we insert (2.4) into (2.2) and arrive to a single second order ODE

$$(hq)'' + \omega^2 q = \frac{\alpha^2}{(hq)^3}\,. \tag{2.5}$$

Recall that for this equation, the unknown is $h(x)$, where $q(x)$ is given. The main result of this section is the following.

Theorem 2.1. *There exists a solution $h \in C_T^+$ of* (2.5) *for any $\alpha \neq 0$, $\omega \neq 0$.*

Proof. By introducing the change of variables $y = hq$ into (2.5), we get the equation

$$y'' + \omega^2 q = \frac{\alpha^2}{y^3}\,.$$

Now, the result is a direct consequence of [8, Theorem 3.12]. □

3 The inverse problem for surface elevation

This section is devoted to studying the inverse problem for the surface elevation. Given a fixed $q \in C_T^+$, the problem is to find $h \in C_T^+$ such that Eq. (1.1) has a travelling wave of the form

$$\eta(x,t) = q(x) \exp i\big[\omega t - \Psi(x)\big]. \tag{3.1}$$

Following the steps of the previous section, we insert (3.1) into (1.1) and separate real and imaginary parts to obtain

$$(hq')' + \omega^2 q - hq\Psi'^2 = 0, \tag{3.2}$$

$$(hq\Psi')' + hq'\Psi' = 0. \tag{3.3}$$

Now, the conserved quantity (energy flux) coming from (3.3) is

$$h(x)q(x)^2\Psi'(x) = \alpha, \quad \forall\, x \in \mathbb{R}.$$

Using this information in (3.2), we arrive to the equation

$$(hq')' + \omega^2 q = \frac{\alpha^2}{hq^3}\,. \tag{3.4}$$

Again, the unknown is h and q is a prescribed function. Although this equation may look similar to (2.5), they are indeed totally different. The fundamental difference is that now we have a first-order differential equation, with the difficulty that q' will change its sign, hence we are dealing with a differential equation that is singular not only in the dependent variable h but also in the independent variable x.

Theorem 3.1. *Let us assume that q is a T-periodic and positive function of class C^2 with a finite number of critical points in $[0,T]$, all of them non-degenerate, that is, if $q'(x) = 0$, then $q''(x) \neq 0$. Then, there exists a threshold $\lambda_0 > 0$ such that*

(i) *there exists a positive T-periodic solution h of (3.4) provided $0 < |\frac{\alpha}{\omega^2}| < \lambda_0$,*

(ii) *no positive T-periodic solution of (3.4) exists provided $|\frac{\alpha}{\omega^2}| > \lambda_0$.*

Moreover,

$$\frac{q_*^5}{4|q_0|} < \lambda_0^2 \leq \min\Big\{ \frac{q^5(b)}{4|q''(b)|} : \;\; q'(b) = 0, q''(b) < 0 \Big\},$$

where

$$q_* \overset{def}{=} \min\big\{q(x) : \; x \in [0,T]\big\}, \quad q_0 \overset{def}{=} \min\big\{q''(x) : x \in [0,T]\big\}.$$

3.1 Sketch of Proof

We assume that q is a T-periodic and positive function of class C^2 with a finite number of critical points in $[0,T]$, all of them non-degenerate, that is, if $q'(x) = 0$, then $q''(x) \neq 0$. Under this assumption, we can divide the interval $[0,T]$ into subintervals $[a,b]$ such that q' is of a constant sign on (a,b) and $q'(a) = q'(b) = 0$. Then, the substitution

$$u(x) = \frac{(h(x)q'(x))^2}{2\omega^4} \;\text{ for }\; x \in (a,b) \tag{3.5}$$

transforms (3.4) into the equation

$$u'(x)=\frac{\lambda^2 q'(x)}{q^3(x)}-q(x)\operatorname{sgn}(q')\sqrt{2u(x)}\ \text{ for } x\in(a,b), \tag{3.6}$$

where $\lambda=\alpha/\omega^2$.

At first we consider an interval $[a,b]$ where

$$q'(a)=0,\quad q'(b)=0,\quad q'(x)>0 \ \text{ for } x\in(a,b),\quad q''(a)>0,\quad q''(b)<0. \tag{3.7}$$

In such an interval, eq. (3.6) reads

$$u'(x)=\frac{\lambda^2 q'(x)}{q^3(x)}-q(x)\sqrt{2u(x)}\ \text{ for } x\in(a,b). \tag{3.8}$$

For technical reasons, we embed this equation into

$$u'(x)=\frac{\lambda^2 q'(x)}{q^3(x)}-q(x)\sqrt{2|u(x)|}\operatorname{sgn}u(x)\ \text{ for } x\in[a,b]. \tag{3.9}$$

Obviously, non-negative solutions of (3.8) and (3.9) are the same.

A solution of (3.9) is understood in the classical sense, that is, a function $u\in C^1([a,b];\mathbb{R})$ satisfying (3.9) for every $x\in[a,b]$. We will investigate the properties of a solution to (3.9) subject to the initial condition

$$u(a)=0. \tag{3.10}$$

Lemma 3.1. *There exists a unique solution u of the initial value problem* (3.9), (3.10). *Moreover, if $\lambda\neq 0$, then*

$$u(x)>0\ \text{ for } x\in(a,b).$$

Lemma 3.2. *Let $\lambda\neq 0$ and let u be the solution to* (3.9), (3.10). *Then, there exists one-sided limit*

$$\ell_a\stackrel{def}{=}\lim_{x\to a+}\frac{\sqrt{2u(x)}}{q'(x)}, \tag{3.11}$$

it is finite, and ℓ_a is the unique positive root of the quadratic equation

$$x^2+\frac{q(a)}{q''(a)}\,x-\frac{\lambda^2}{q^3(a)q''(a)}=0. \tag{3.12}$$

Lemma 3.3. *Let u be a solution of* (3.9) *satisfying*

$$u(x)>0\ \text{ for } x\in(x_0,b)$$

for some $x_0\in(a,b)$ and $u(b)=0$. Then, there exists one-sided limit

$$\ell_b\stackrel{def}{=}\lim_{x\to b-}\frac{\sqrt{2u(x)}}{q'(x)}, \tag{3.13}$$

it is finite, and ℓ_b is a root of the quadratic equation

$$x^2-\frac{q(b)}{|q''(b)|}\,x+\frac{\lambda^2}{q^3(b)|q''(b)|}=0. \tag{3.14}$$

Lemma 3.4. *There exists a threshold $\lambda_{ab}>0$ such that*

(i) *if* $0 < |\lambda| < \lambda_{ab}$*, the unique solution* u *of* (3.9), (3.10) *satisfies* $u(b) = 0$*. Moreover,* ℓ_a, ℓ_b *defined by* (3.11) *and* (3.13) *are respectively the unique positive root of* (3.12) *and the smaller root of* (3.14)*;*

(ii) *if* $|\lambda| = \lambda_{ab}$*, the unique solution* u *of* (3.9), (3.10) *satisfies* $u(b) = 0$*. Moreover,* ℓ_a, ℓ_b *defined by* (3.11) *and* (3.13) *are respectively the unique positive root of* (3.12) *and a root of* (3.14)*;*

(iii) *if* $|\lambda| > \lambda_{ab}$*, the unique solution* u *of* (3.9), (3.10) *satisfies* $u(b) > 0$*.*

The case when

$$q'(a) = 0, \quad q'(b) = 0, \quad q'(x) < 0 \text{ for } x \in (a,b), \quad q''(a) < 0, \quad q''(b) > 0$$

can be transformed by $\widetilde{q}(x) = q(-x)$ to the previous case.

The threshold λ_0 is then a minimum of the thresholds λ_{ab} that correspond to each subinterval (a,b). Further, the change (3.5) is inverted by defining

$$h(x) = \frac{\omega^2\sqrt{2u(x)}}{|q'(x)|} \text{ for } x \in (a,b), \quad h(a) = \omega^2\ell_a, \quad h(b) = \omega^2\ell_b,$$

on every subinterval (a,b). By construction, h is a positive absolutely continuous T-periodic function.

3.2 Estimation of the threshold λ_0

Theorem 3.1 includes a general quantitative estimate of the threshold value λ_0. In this subsection, we develop a technique that permits a significant improvement of the estimates in concrete examples. Since λ_0 is the minimum of the thresholds λ_{ab} corresponding to each subinterval (a,b), we only focus on estimating the latter. As in the previous subsection we formulate the results for the case when (3.7) is valid.

Theorem 3.2. *Let there exist positive constants* λ_1 *and* λ_2 *such that* $\lambda_1 \le \lambda_2$*, and let* $v, w \in AC([a,b];\mathbb{R})$ *satisfy*

$$v'(x) \ge \frac{\lambda_1^2 q'(x)}{q^3(x)} - q(x)\sqrt{2|v(x)|}\,\operatorname{sgn} v(x) \text{ for a.e. } x \in [a,b],$$
$$w'(x) \le \frac{\lambda_2^2 q'(x)}{q^3(x)} - q(x)\sqrt{2|w(x)|}\,\operatorname{sgn} w(x) \text{ for a.e. } x \in [a,b],$$
$$v(a) \ge 0 \ge w(a), \quad v(b) = 0 = w(b),$$
$$\liminf_{x\to b-} \frac{\sqrt{2|w(x)|}\,\operatorname{sgn} w(x)}{q'(x)} > x_1(\lambda_2),$$

where $x_1(\lambda_2)$ *is the smaller root of* (3.14) *with* $\lambda = \lambda_2$*. Then, the threshold* λ_{ab} *admits the estimate*

$$\lambda_1 \le \lambda_{ab} \le \lambda_2. \tag{3.15}$$

If we put

$$v(x) \overset{def}{=} \frac{(\ell_1(x)q'(x))^2}{2}, \quad w(x) \overset{def}{=} \frac{(\ell_2(x)q'(x))^2}{2} \text{ for } x \in [a,b],$$

then Theorem 3.2 yields the following assertion.

Corollary 3.1. *Let there exist positive constants λ_1 and λ_2 such that $\lambda_1 \le \lambda_2$, let $q \in C^2([a,b];\mathbb{R})$, and let $\ell_1, \ell_2 \in C^1([a,b];\mathbb{R})$ satisfy*

$$\ell_i(x) > 0 \ \text{ for } \ x \in [a,b] \ \ (i = 1,2), \tag{3.16}$$

$$\ell_1(x)\big(\ell_1'(x)q'(x) + \ell_1(x)q''(x)\big) \ge \frac{\lambda_1^2}{q^3(x)} - q(x)\ell_1(x) \ \text{ for } \ x \in [a,b], \tag{3.17}$$

$$\ell_2(x)\big(\ell_2'(x)q'(x) + \ell_2(x)q''(x)\big) \le \frac{\lambda_2^2}{q^3(x)} - q(x)\ell_2(x) \ \text{ for } \ x \in [a,b], \tag{3.18}$$

$$\ell_2(b) > x_1(\lambda_2), \tag{3.19}$$

where $x_1(\lambda_2)$ is the smaller root of (3.14) *with $\lambda = \lambda_2$. Then, the threshold λ_{ab} admits the estimate* (3.15)*.*

3.3 A concrete example

Theorem 3.1 provides a general quantitative estimate for the threshold value λ_0. However, such an estimate can be improved for particular cases by a suitable construction of upper and lower functions. To illustrate this idea, consider $q(x) = 2 - \cos x$ for $x \in [0, 2\pi]$. Then local extremes of q divide the interval $[0, 2\pi]$ into two subintervals, in particular, we set $T = 2\pi$, $x_1 = 0$, $x_2 = \pi$, $x_1 + T = 2\pi$ in order to apply Theorem 3.1. Then we have

$$q'(x) > 0 \ \text{ for } \ x \in (0,\pi), \quad q'(x) < 0 \ \text{ for } \ x \in (\pi, 2\pi),$$
$$q'(0) = q'(\pi) = q'(2\pi) = 0, \quad q''(0) = q''(2\pi) = 1, \quad q''(\pi) = -1.$$

Moreover, since q is symmetric with respect to π, we can easily conclude that the thresholds corresponding to each subinterval has the same value, i.e., $\lambda_0 = \lambda_{0,\pi} = \lambda_{\pi,2\pi}$. Thus, according to Theorem 3.1, the threshold λ_0 satisfies the inequalities

$$0.25 = \frac{1}{4} < \lambda_0^2 \le \frac{243}{4} = 60.75.$$

Let us see how to improve this estimate by constructing a specific couple of upper and lower functions.

According to Corollary 3.1, it is sufficient to find suitable functions $\ell_1(x)$ and $\ell_2(x)$ that satisfy (3.16)–(3.19). Obviously, we can start with positive constant functions. Then, if we put

$$\lambda_1^2 \overset{def}{=} \min\big\{(\ell_1^2 q''(x) + \ell_1 q(x))q^3(x) : \ \ x \in [0,\pi]\big\},$$
$$\lambda_2^2 \overset{def}{=} \max\big\{(\ell_2^2 q''(x) + \ell_2 q(x))q^3(x) : \ \ x \in [0,\pi]\big\},$$

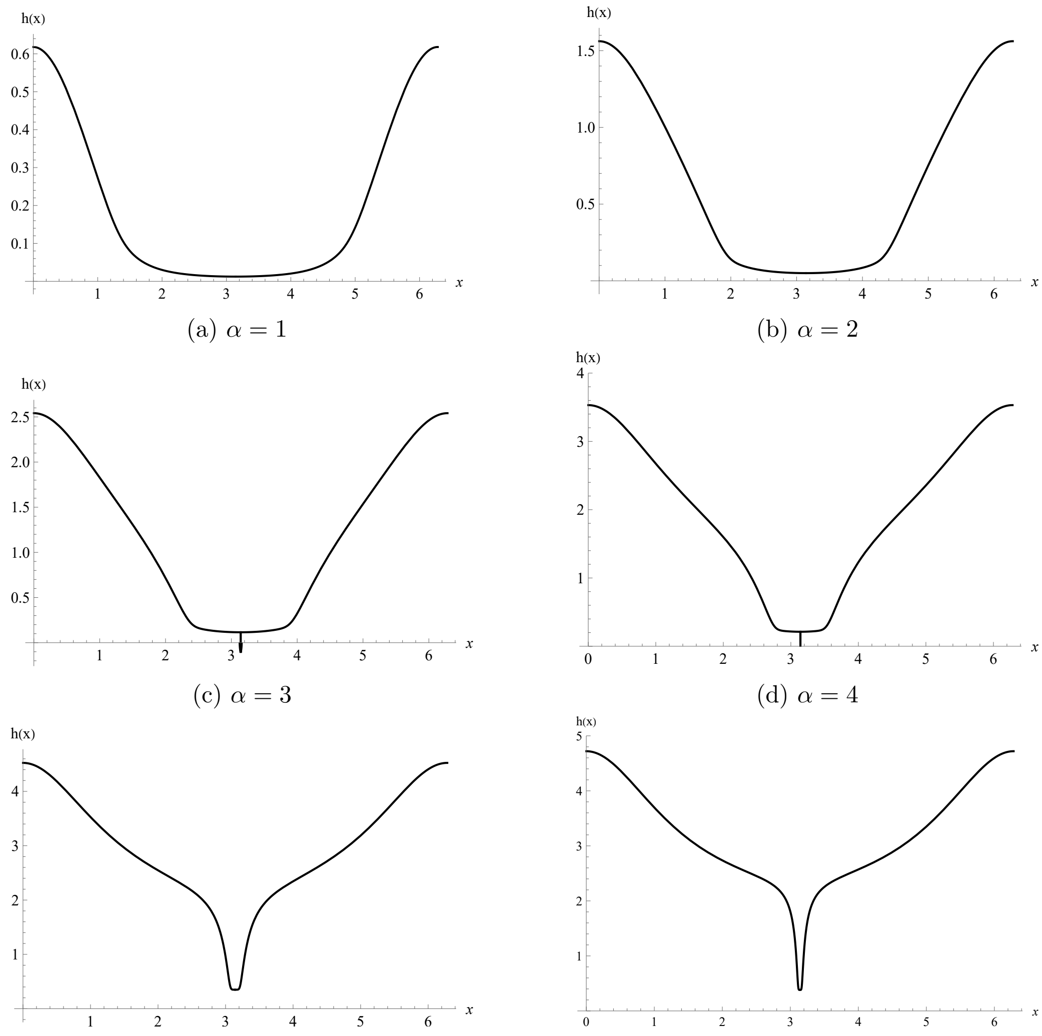
we can easily verify that inequalities (3.17) and (3.18) with $a = 0$, $b = \pi$ are fulfilled. Consequently, if also (3.19) is fulfilled, then we can conclude that (3.15) holds.

Analyzing the function $x \mapsto (\ell^2 q''(x) + \ell q(x))q^3(x)$ in detail, one can show that the optimal values for constant functions ℓ_1 and ℓ_2 are

$$\ell_1 = \frac{20}{7}\,, \quad \ell_2 = \frac{5}{2}\,.$$

Then, we get

$$\lambda_1^2 = \frac{540}{49} \approx 11.020408163, \quad \lambda_2^2 = \frac{3125}{64} = 48.828125, \qquad x_1(\lambda_2) \approx 0.835507015894,$$



(a) $\alpha = 1$ (b) $\alpha = 2$

(c) $\alpha = 3$ (d) $\alpha = 4$

(e) $\alpha = 5$ (f) $\alpha = \sqrt{27}$

Figure 2. Numerically computed water depth function $h(x)$ for an amplitude $q(x) = 2 - \cos x$ of the travelling wave. We fixed $\omega = 1$ and moved the energy flux parameter α.

and we have the estimate

$$\frac{540}{49} \le \lambda_0^2 \le \frac{3125}{64}\,.$$

Let us pass to nonconstant functions $\ell_1(x)$ and $\ell_2(x)$. Then the choice

$$\ell_i(x) \overset{def}{=} a_i + b_i \cos x + c_i \sin x + d_i \sin x \cos x \quad \text{for} \quad x \in [0, \pi] \quad (i = 1, 2),$$

where

$$a_1 = 4.265, \quad b_1 = 1.639, \quad c_1 = -1.075, \quad d_1 = -0.778,$$
$$a_2 = 3.605, \quad b_2 = 1.025, \quad c_2 = -0.408, \quad d_2 = -0.222,$$

guarantees that $\ell_1(x)$ and $\ell_2(x)$ satisfy (3.16)–(3.19) with $a = 0$, $b = \pi$, $\lambda_1^2 = 26.4$, and $\lambda_2^2 = 31.68$. Furthermore, note also that

$$\ell_1(\pi) < x_2(\lambda_1), \quad x_2(\lambda_2) < \ell_2(\pi), \tag{3.20}$$

where $x_2(\lambda_i)$ is the greater root of (3.14) with $\lambda = \lambda_i$ $(i = 1, 2)$. Indeed,

$$\ell_1(\pi) = 2.626, \quad x_2(\lambda_1) \approx 2.62792828771, \quad x_2(\lambda_2) \approx 2.53762549442, \quad \ell_2(\pi) = 2.58.$$

The condition (3.20) is stronger than (3.19) and allows strict inequalities in the threshold estimate. Therefore, according to Corollary 3.1 we have

$$26.4 < \lambda_0^2 < 31.68.$$

We conducted several numerical calculations to approximately solve the relevant equations and determine the water depth function $h(x)$ associated with the amplitude $q(x) = 2 - \cos x$. The results are illustrated in Fig. 2. Notably, as λ approaches the critical value λ_0, a singularity arises in $h(x)$.

References

[1] G. Bluman and S. Kumei, On invariance properties of the wave equation. *J. Math. Phys.* **28** (1987), no. 2, 307–318.

[2] S. Churilov and Yu. Stepanyants, Long wave propagation in canals with spatially varying cross-sections and currents. *Nonlinear and modern mathematical physics*, 19–79, Springer Proc. Math. Stat., 459, *Springer, Cham*, 2024.

[3] D. L. Clements and C. Rogers, Analytic solution of the linearized shallow-water wave equations for certain continuous depth variations. *J. Austral. Math. Soc. Ser. B* **19** (1975), no. 1, 81–94.

[4] I. Didenkulovaa, E. Pelinovsky and Tarmo Soomere, Exact travelling wave solutions in strongly inhomogeneous media. *Estonian Journal of Engineering* **14** (2008), no. 3, 220–231.

[5] I. I. Didenkulova, N. Zahibo and E. N. Pelinovsky, Reflection of long waves from a "nonreflecting" bottom profile. (Russian) *Izv. Ross. Akad. Nauk Mekh. Zhidk. Gaza* **2008**, no. 4, 102–108; translation in *Fluid Dyn.* **43** (2008), no. 4, 590–595.

[6] R. Grimshaw, D. Pelinovsky and E. Pelinovsky, Homogenization of the variable-speed wave equation. *Wave Motion* **47** (2010), no. 8, 496–507.

[7] R. Hernández-Walls, B. Martín-Atienza, M. Salinas-Matus and J. Castillo, Solving the linear inviscid shallow water equations in one dimension, with variable depth, using a recursion formula. *Eur. J. Phys.* **38** (2017), Paper no. 065802.

[8] A. C. Lazer and S. Solimini, On periodic solutions of nonlinear differential equations with singularities. *Proc. Amer. Math. Soc.* **99** (1987), no. 1, 109–114.

[9] E. Pelinovsky and O. Kaptsov, Traveling waves in shallow seas of variable depths. *Symmetry* **14** (2022), Paper no. 1448, no. 7.

[10] E. N. Pelinovsky and O. V. Kaptsov, Traveling waves in nondispersive strongly inhomogeneous media. *Dokl. Phys.* **67** (2022), 415–419.