Asymptotic Behaviour of Rapidly Changing Solutions of Fourth-Order Differential Equation with Rapidly Changing Nonlinearity

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The following differential equation is considered

$$y^{(4)} = \alpha_0 p_0(t)\varphi(y), \tag{1}$$

where $\alpha_0 \in \{-1,1\}$, $p_0 : [a,\omega[\to]0,+\infty[$ – continuous function, $-\infty < a < \omega \le +\infty$, $\varphi : \Delta_{Y_0} \to]0,+\infty[$ – is a twice continuously differentiable function such that

$$\varphi'(y) \ne 0 \ \text{ at } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \varphi(y) = \begin{cases} \text{either} & 0, \\ \text{or} & +\infty, \end{cases} \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\varphi(y)\varphi''(y)}{\varphi'^2(y)} = 1, \tag{2}$$

Y_0 is equal to either 0, or $\pm\infty$, Δ_{Y_0} – unilateral dislocation Y_0. It directly follows from condition (2) that

$$\frac{\varphi'(y)}{\varphi(y)} \sim \frac{\varphi''(y)}{\varphi'(y)} \ \text{ as } \ y \to Y_0 \ \ (y \in \Delta_{Y_0}) \ \text{ and } \ \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{y\varphi'(y)}{\varphi(y)} = \pm\infty.$$

According to these conditions, the function φ and its first-order derivative (see the monograph by M. Maric [5, Chapter 3, Section 3.4, Lemmas 3.2, 3.3, pp. 91–92]) are rapidly changing functions as $y \to Y_0$.

Definition 1. A solution y of the differential equation (1) is called $P_\omega(Y_0,\lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if it is defined on the interval $[t_0,\omega[\subset [a,\omega[$ and satisfies the following conditions

$$y(t) \in \Delta_{Y_0} \ \text{ at } \ t \in [t_0,\omega[\,, \quad \lim_{t\uparrow\omega} y(t) = Y_0,$$

$$\lim_{t\uparrow\omega} y^{(k)}(t) = \begin{cases} \text{either } \ 0, \\ \text{or } \ \pm\infty, \end{cases} \quad (k=1,2,3), \quad \lim_{t\uparrow\omega} \frac{[y'''(t)]^2}{y''(t)y^{(4)}(t)} = \lambda_0.$$

Earlier, the asymptotic behaviour of $P_\omega(Y_0,\lambda_0)$-solutions of equation (1) was investigated in the case when $\lambda_0 \in \mathbb{R}\setminus\{0,\frac12,\frac23,1\}$ in [3]. The purpose of this paper is to study the existence and asymptotic behaviour of $P_\omega(Y_0,\lambda_0)$-solutions in the special case when $\lambda_0 = 1$. In this case, due to the a priori asymptotic properties of the $P_\omega(Y_0,\lambda_0)$-solutions (see [1, Section 3, Subsection 10]), the following asymptotic relations hold for each $P_\omega(Y_0,1)$-solution

$$\frac{y'(t)}{y(t)} \sim \frac{y''(t)}{y'(t)} \sim \frac{y'''(t)}{y''(t)} \sim \frac{y''''(t)}{y'''(t)} \ \text{ at } \ t\uparrow\omega, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)y'(t)}{y(t)} = \pm\infty,$$

where

$$\pi_\omega(t) = \begin{cases} t, & \text{or } \ \omega = +\infty, \\ t-\omega, & \text{or } \ \omega < +\infty. \end{cases}$$

It follows, in particular, that the $P_\omega(Y_0,1)$-solution of equation (1) and its derivatives up to and including the third order are rapidly changing functions at $t\uparrow\omega$.

Let us introduce the necessary auxiliary notation and assume that the domain of the function φ in equation (1) is defined as follows

$$\Delta_{Y_0}=\Delta_{Y_0}(y_0),\ \ \text{where}\ \ \Delta_{Y_0}(y_0)=\begin{cases}[y_0,Y_0[& \text{if}\ \ \Delta_{Y_0}\ \text{ left neighbourhood } Y_0,\\ ]Y_0,y_0] & \text{if}\ \ \Delta_{Y_0}\ \text{ right neighbourhood } Y_0,\end{cases}$$

and $y_0\in\Delta_{Y_0}$ such that $|y_0|<1$ at $Y_0=0$ and $y_0>1$ ($y_0<-1$) at $Y_0=+\infty$ (at $Y_0=-\infty$).

Next, let's assume that

$$\mu_0=\operatorname{sign}\varphi'(y),\quad \nu_0=\operatorname{sign}y_0,\quad \nu_1=\begin{cases}1 & \text{if}\ \ \Delta_{Y_0}=[y_0,Y_0[\,,\\ -1 & \text{if}\ \ \Delta_{Y_0}=]Y_0,y_0],\end{cases}$$

and introduce the following functions

$$J_0(t)=\int\limits_{A_0}^{t}p^{\frac14}(\tau)\,d\tau,\quad \Phi(y)=\int\limits_{B}^{y}\frac{ds}{|s|^{\frac34}\varphi^{\frac14}(s)}\,,$$

where $p:[a,\omega[\,\to]0,+\infty[$ is a continuous or continuously differentiable function as $t\uparrow\omega$,

$$A_0=\begin{cases}\omega & \text{if}\ \displaystyle\int\limits_a^\omega p^{\frac14}(\tau)\,d\tau<+\infty,\\ a & \text{if}\ \displaystyle\int\limits_a^\omega p^{\frac14}(\tau)\,d\tau=+\infty,\end{cases}\qquad B=\begin{cases}Y_0 & \text{if}\ \displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{|s|^{\frac34}\varphi^{\frac14}(s)}=const,\\ y_0 & \text{if}\ \displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{|s|^{\frac34}\varphi^{\frac14}(s)}=\pm\infty.\end{cases}$$

Let's pay attention to some properties of the function Φ. It keeps its sign at Δ_{Y_0}, goes either to zero or to $\pm\infty$ at $y\to Y_0$, and increases at Δ_{Y_0}, since in this interval $\Phi'(t)=|y|^{-\frac34}\varphi^{-\frac14}(y)>0$. Therefore, there exists an inverse function $\Phi^{-1}:\Delta_{Z_0}\to\Delta_{Y_0}$, where, due to the second condition (2) and the monotonic growth of Φ^{-1},

$$Z_0=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\Phi(y)=\begin{cases}\text{either} & 0,\\ \text{or} & \pm\infty,\end{cases}$$

$$\Delta_{Z_0}=\begin{cases}[z_0,Z_0[& \text{if}\ \ \Delta_{Y_0}=[y_0,Y_0[\,,\\ ]Z_0,z_0] & \text{if}\ \ \Delta_{Y_0}=]Y_0,y_0],\end{cases}\qquad z_0=\Phi(y_0),$$

Given Definition 1, we note that the numbers ν_0 and ν_1 determine the signs of any $P_\omega(Y_0,1)$-solution and the first derivative in some left neighbourhood of ω. The conditions

$$\nu_0\nu_1<0,\ \text{ if }\ Y_0=0,\quad \nu_0\nu_1>0,\ \text{ if }\ Y_0=\pm\infty,$$

are necessary for the existence of such solutions.

In addition to the above designations, we will also introduce auxiliary functions:

$$H(t)=\frac{\Phi^{-1}(\nu_1J_0(t))\varphi'(\Phi^{-1}(\nu_1J_0(t)))}{\varphi(\Phi^{-1}(\nu_1J_0(t)))}\,,$$

$$J_1(t)=\int\limits_{A_1}^{t}p(\tau)\varphi(\Phi^{-1}(\nu_1J_0(\tau)))\,d\tau,\quad J_2(t)=\int\limits_{A_2}^{t}J_1(\tau)\,d\tau,\quad J_3(t)=\int\limits_{A_3}^{t}J_2(\tau)\,d\tau,$$

where

$$A_1 = \begin{cases} t_1 & \text{if} \displaystyle\int_{t_1}^{\omega} p(\tau)\varphi(\Phi^{-1}(\nu_1 J_0(\tau)))\,d\tau = +\infty, \\ \omega & \text{if} \displaystyle\int_{t_1}^{\omega} p(\tau)\varphi(\Phi^{-1}(\nu_1 J_0(\tau)))\,d\tau < +\infty, \end{cases} \qquad t_1 \in [a,\omega],$$

$$A_2 = \begin{cases} t_1 & \text{if} \displaystyle\int_{t_1}^{\omega} J_1(\tau)\,d\tau = +\infty, \\ \omega & \text{if} \displaystyle\int_{t_1}^{\omega} J_1(\tau)\,d\tau < +\infty, \end{cases} \qquad A_3 = \begin{cases} t_1 & \text{if} \displaystyle\int_{t_1}^{\omega} J_2(\tau)\,d\tau = +\infty, \\ \omega & \text{if} \displaystyle\int_{t_1}^{\omega} J_2(\tau)\,d\tau < +\infty, \end{cases}$$

The following statement is true for equation (1).

Theorem. *For the existence of $P_\omega(Y_0,1)$-solutions of the differential equation* (1)*, the following inequalities must be satisfied*

$$\nu_1\mu_0 J_0(t) < 0 \ \ \textit{at} \ \ t \in]a,\omega[\,,$$
$$\alpha_0\nu_1 < 0 \ \ \textit{if} \ \ Y_0 = 0, \quad \alpha_0\nu_1 > 0 \ \ \textit{if} \ \ Y_0 = \pm\infty,$$

and conditions

$$\frac{\alpha_0 J_3(t)}{\Phi^{-1}(\nu_1 J_0(t))} \sim \frac{J_1'(t)}{J_1(t)} \sim \frac{J_2'(t)}{J_2(t)} \sim \frac{J_3'(t)}{J_3(t)} \sim \frac{(\Phi^{-1}(\nu_1 J_0(t)))'}{\Phi^{-1}(\nu_1 J_0(t))} \ \ \textit{as} \ \ t \uparrow \omega,$$

$$\lim_{t\uparrow\omega} H(t) = \pm\infty, \quad \nu_1 \lim_{t\uparrow\omega} J_0(t) = Z_0, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)(\Phi^{-1}(\nu_1 J_0(t)))'}{\Phi^{-1}(\nu_1 J_0(t)))} = \pm\infty, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t) J_0'(t)}{J_0(t)} = \pm\infty.$$

In addition, for each such solution, the following asymptotic representations are obtained

$$y(t) = \Phi^{-1}(\nu_1 J_0(t))\Big[1 + \frac{o(1)}{H(t)}\Big],$$
$$y'(t) = \alpha_0 J_3(t)[1+o(1)], \quad y''(t) = \alpha_0 J_2(t)[1+o(1)], \quad y'''(t) = \alpha_0 J_1(t)[1+o(1)] \ \ \textit{as} \ \ t \uparrow \omega,$$

The sufficiency of the obtained conditions is proved by imposing an additional condition. Namely,

$$\lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \frac{\left(\frac{\varphi'(y)}{\varphi(y)}\right)'}{\left(\frac{\varphi'(y)}{\varphi(y)}\right)^2} \left|\frac{y\varphi'(y)}{\varphi(y)}\right|^{\frac{3}{4}} = 0.$$

The question of the actual existence of solutions with these asymptotic images is established under some additional conditions, by transforming the found asymptotic images to a system of quasilinear equations using Theorem 2.2 from the work by Evtukhov V. M., Samoilenko A. M. [4] on the existence of solutions tending to zero.

References

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