On the Stability Conditions of a Nonlinear Biological Epidemic Model

D. A. Gabidullin

Lomonosov Moscow State University, Moscow, Russia
E-mail: david0166@yandex.ru

We consider the dynamical system

$$\begin{cases} \dfrac{dS}{dt} = (1-p)a - dS - \dfrac{\beta IS}{1+\sigma I^k} + \delta V, \\ \dfrac{dE}{dt} = \dfrac{\beta IS}{1+\sigma I^k} - (d+\varepsilon+\eta)E, \\ \dfrac{dI}{dt} = \varepsilon E - (d+\tau)I, \\ \dfrac{dV}{dt} = pa + \tau I + \eta E - (d+\delta)V, \end{cases} \tag{1}$$

which arises in an epidemiological model incorporating an incubation period and temporary immunity. The population N is divided into four categories: susceptible (S), exposed (E), infected (I), and vaccinated/recovered (V). All parameters of system (1) are non-negative, and their biological meanings are interpreted as follows: individuals are born at a rate a and enter the susceptible class S, while a fraction of newborns is effectively vaccinated at a rate p. Susceptible individuals become infected at a rate β. Temporary immunity (caused by an ideal vaccine, disease, or asymptomatic infections) wanes at a rate δ. All individuals in every class experience the same natural mortality rate d. Individuals in the exposed class E can transition to the infected class I at a rate ε, as well as to the vaccinated/recovered class V at a rate η (due to the acquisition of natural immunity). Infected individuals effectively recover at a rate τ, and the parameters σ and k will be described below.

The $SEIVS$ and $SIRS$ models with various incidence rates have been studied in papers [1–3, 5–8, 10]: in [2, 5, 6, 8], the $SEIVS$ models were analyzed using a geometric approach to establish asymptotic stability and global asymptoticity of equilibrium states depending on the control reproduction number R_c. In [7], the geometric criterion for global asymptoticity was generalized. In [1, 3], diffusion effects of epidemic spread in a population were considered for the $SIRS$ models. In [10], an SIR model with a specific type of infectious force was examined.

Below, a new infectious force is considered

$$\varphi(I) := \frac{\beta I}{1+\sigma I^k}\,,$$

where the parameters σ and k account for inhibitory or psychological effects caused by public. System (1) has an equilibrium point for any parameter values given by

$$Q_0 = (S_0, 0, 0, V_0), \quad S_0 \equiv \frac{a((1-p)d+\delta)}{d(d+\delta)}\,, \quad V_0 \equiv \frac{pa}{d+\delta}\,,$$

which corresponds to the absence of infected individuals in the population.

System (1) admits a biologically feasible region

$$\mathcal{D} = \Big\{(S,E,I,V) \in \mathbb{R}^4_+ : \;\; S \leqslant S_0, \;\; V \leqslant V_0, \;\; E \leqslant S_0 + V_0, \;\; I \leqslant S_0 + V_0, \;\; S+E+I+V \leqslant \frac{a}{d}\Big\},$$

which is positively invariant. The control reproduction number, depending on the parameters of the model, is defined as

$$R_c := \frac{\varepsilon S_0 \varphi'(0)}{(d+\tau)(d+\epsilon+\eta)}\,.$$

Definition. An equilibrium point x^* is called:

- *asymptotically stable* if all solutions starting sufficiently close to it not only remain near x^* for all time but also converge to x^* as time tends to infinity;

- *globally asymptotically stable* if every solution, regardless of the initial condition, converges to x^* as time tends to infinity.

Theorem 1. *The equilibrium point Q_0 of system* (1) *is globally asymptotically stable if $R_c \leqslant 1$ and unstable if $R_c > 1$.*

Proof. The local stability of Q_0 is established using the next-generation operator method developed in [9]. Using the notation from [9], the matrices F and V for the model take the form

$$F = \begin{bmatrix} 0 & S_0\varphi'(0) \\ 0 & 0 \end{bmatrix}, \quad V = \begin{bmatrix} d+\varepsilon+\eta & 0 \\ -\varepsilon & d+\tau \end{bmatrix},$$

so that the control reproduction number for the model is given by

$$R_c = \rho(FV^{-1}) = \frac{\varepsilon S_0 \varphi'(0)}{(d+\tau)(d+\varepsilon+\eta)}\,,$$

where ρ is the spectral radius of the matrix.

Following Theorem 2 in [9], we obtain the first part of the theorem statement.

The Lyapunov function is $V(t) = E + \frac{(d+\varepsilon+\eta)I}{\varepsilon}$. Since $\varphi'(I) \leq \frac{\varphi(I)}{I}$, this indicates the monotonic non-increase of $\frac{\varphi(I)}{I}$ for $I > 0$, so that

$$\frac{\varphi(I)}{I} \leq \lim_{I \to 0^+} \frac{\varphi(I)}{I} = \varphi'(0).$$

Along the trajectories of system (1), the time derivative of $V(t)$ can be computed as

$$\begin{aligned} \frac{dV(t)}{dt} &= I\Big(S\frac{\varphi(I)}{I} - \frac{(d+\tau)(d+\varepsilon+\eta)}{\varepsilon}\Big) \\ &\leq I\Big(S_0\varphi'(0) - \frac{(d+\tau)(d+\varepsilon+\eta)}{\varepsilon}\Big) = (R_c - 1)\,\frac{(d+\tau)(d+\varepsilon+\eta)}{\varepsilon}\,I \leq 0. \end{aligned}$$

Therefore, by LaSalle's invariance principle [4] and the local stability of Q_0, it follows that Q_0 is globally asymptotically stable in $\mathcal{D}$ when $R_c \leq 1$. □

Theorem 2. *If $R_c > 1$, then system* (1) *has another equilibrium point Q^* in the region $\mathcal{D}$, distinct from Q_0, which is asymptotically stable.*

Proof. For system (1), the coordinates of the positive equilibrium are determined as follows:

$$\begin{cases} 0 = (1-p)A - dS - S\varphi(I) + \delta V, \\ 0 = S\varphi(I) - (d+\varepsilon+\eta)E, \\ 0 = \varepsilon E - (d+\tau)I, \\ 0 = pA + \tau I + \eta E - (d+\delta)V. \end{cases} \tag{2}$$

For convenience, let

$$\theta_1 := \frac{(d+\tau)(d+\varepsilon+\eta)}{\varepsilon} \ \text{ and } \ \theta_2 := \frac{(d+\tau)(d+\varepsilon+\eta+\delta)+\varepsilon\delta}{\varepsilon(d+\delta)}\,.$$

Using the third equation in (2), we obtain $E = \frac{(d+\tau)I}{\varepsilon}$. Adding the first two equations and using the last equation, we get

$$\begin{gathered} (1-p)A - dS - \theta_1 I + \delta V = 0, \\ pA + \tau\varepsilon + \frac{(d+\tau)\eta}{\varepsilon}\, I - (d+\delta)V = 0. \end{gathered} \tag{3}$$

Eliminating V from these equations leads to $S = S_0 - \theta_2 I$. Given $S \ge 0$, we obtain $I \le S_0/\theta_2$. Using the second equation in (2), we obtain

$$\Phi(I) := (S_0 - \theta_2 I)\varphi(I) - \theta_1 I = 0, \ \ 0 < I \le \frac{S_0}{\theta_2}\,. \tag{4}$$

The existence and uniqueness of the positive solution to equation (4) proceed in the following three steps.

Step 1. Existence of a positive solution for $R_c > 1$. In fact, from

$$\Phi'(I) = -\theta_2\varphi(I) + (S_0 - \theta_2 I)\varphi'(I) - \theta_1$$

and since $\varphi(0) = 0$, we have

$$\Phi'(0) = \lim_{I\to 0^+} S_0\varphi'(I) - \theta_1 = \theta_1(R_c - 1),$$

which can be achieved. Given $R_c > 1$, it is easy to show that $\Phi(I) > 0$ for sufficiently small values of I, since $\Phi'(0) > 0$, $\Phi(0) = 0$, and $\Phi(S_0/\theta_2) < 0$. This means that at least one positive solution to equation (4) exists. Let us denote this solution by I^*.

Step 2. It can be verified that the positive solution I^* is unique for $R_c > 1$. Without loss of generality, assume that another positive root, closest to I^*, exists and is denoted by $I^\dagger$. Then, the inequality $\Phi'(I^\dagger) \ge 0$ follows from the continuity of $\Phi(I)$. Using the properties of the function φ, we obtain:

$$\Phi'(I^\dagger) = (S^\dagger)\varphi'(I^\dagger) - \theta_2\varphi(I^\dagger) - \frac{(S^\dagger)\varphi(I^\dagger)}{I^\dagger} < 0. \tag{5}$$

This leads to a contradiction and confirms the uniqueness of I^*.

Step 3. We prove the absence of a positive root for (4) in the case $R_c \le 1$ by contradiction. Assume that there exists a smallest positive root I^+. Then, it is evident that $\Phi'(I^+) < 0$ according to (5). Since $\Phi(0) = 0$ and $\Phi'(0) \le 0$, we have $\Phi(I) \le 0$ for sufficiently small values of I. Thus, the continuous function $\Phi(I)$ increases from a non-positive value to 0, which implies that $\Phi'(I^+) \ge 0$, leading to a contradiction.

Therefore, from Steps 1–3, we conclude that model (1) has a unique endemic equilibrium point $Q^* = (S^*, E^*, I^*, V^*)$ if and only if $R_c > 1$, where S^*, E^*, V^* can be uniquely determined according to the results derived above.

The Jacobian matrix of model (1) is given by

$$J = \begin{bmatrix} -(d+\varphi(I)) & 0 & -S\varphi'(I) & \delta \\ \varphi(I) & -(d+\varepsilon+\eta) & S\varphi'(I) & 0 \\ 0 & \varepsilon & -(d+\tau) & 0 \\ 0 & \eta & \tau & -(d+\delta) \end{bmatrix},$$

so that the characteristic equation at the point Q^* is given by

$$(\chi+d)\Big[(\chi+d+\tau)(\chi+d+\varepsilon+\eta)(\chi+d+\delta+\varphi(I^*))+ \\ +\delta\varphi(I^*)(\chi+d+\tau+\varepsilon)-\varepsilon S^*\varphi'(I^*)(\chi+d+\delta)\Big]=0. \quad (6)$$

Clearly, $\chi_1 = -d < 0$. As for the remaining eigenvalues of the equation:

$$(\chi+d+\varepsilon+\eta)(\chi+d+\delta+\varphi(I^*))+\delta\varphi(I^*)(\chi+d+\tau+\varepsilon)=\varepsilon S^*\varphi'(I^*)(\chi+d+\delta). \quad (7)$$

Case I. $\varphi'(I^*) > 0$. It is claimed that all eigenvalues of equation (7) have negative real parts. Otherwise, there exists at least one eigenvalue $\widetilde{\chi}$ such that $\operatorname{Re}\widetilde{\chi} \geq 0$. From this, it follows that

$$(d+\tau)(d+\varepsilon+\eta) \\ < \left|(\widetilde{\chi}+d+\tau)(\widetilde{\chi}+d+\varepsilon+\eta)\Big(1+\frac{\varphi(I^*)}{\widetilde{\chi}+d+\delta}\Big)+\delta\varphi(I^*)\,\frac{\widetilde{\chi}+d+\tau+\varepsilon}{\widetilde{\chi}+d+\delta}\right| \\ = \varepsilon S^*\varphi'(I^*) \leq \frac{\varepsilon S^*\varphi(I^*)}{I^*} = (d+\tau)(d+\varepsilon+\eta). \quad (8)$$

Therefore, each eigenvalue χ of equation (6) satisfies $\operatorname{Re}\chi < 0$.

Case II. $\varphi(I^*) \leq 0$. Equation (7) can be reformulated as $\chi^3 + H_1\chi^2 + H_2\chi + H_3 = 0$, where H_1, H_2, and H_3 are defined by the relations

$$H_1 = h_1+h_2+h_3, \quad H_2 = h_1h_2+h_1h_3+h_2h_3+\delta\varphi(I^*)-\varepsilon S^*\varphi'(I^*), \\ H_3 = h_1h_2h_3+\delta\varphi(I^*)h_4-\varepsilon S^*\varphi'(I^*)h_5,$$

where

$$h_1 = d+\tau, \quad h_2 = d+\varepsilon+\eta, \quad h_3 = d+\delta+\varphi(I^*), \quad h_4 = d+\tau+\varepsilon, \quad h_5 = d+\delta.$$

According to the Routh–Hurwitz stability criterion, the necessary and sufficient conditions for the stability of Q^* are:

(i) $H_i > 0$, $i = 1, 2, 3$;

(ii) $H_1H_2 - H_3 > 0$.

It is evident that (i) holds, as $h_i > 0$. Moreover, (ii) can be guaranteed by

$$H_1H_2 - H_3 = \Big[(h_1+h_2+h_3)(h_1h_2+h_1h_3+h_2h_3)-h_1h_2h_3\Big] \\ +\delta\varphi(I^*)(h_1+h_2+h_3-h_4)-\varepsilon S^*\varphi(I^*)(h_1+h_2+h_3-h_5) > 0.$$

By combining cases I and II, it can be concluded that Q^* is locally asymptotically stable if and only if $R_c > 1$. □

References

[1] I. Astashova, V. Chebotaeva and A. Cherepanov, Mathematical models of epidemics in closed populations and their visualization via web application phapl. *WSEAS Transactions on Biology and Biomedicine* **15** (2018), no. 12, 112–118.

[2] L.-M. Cai and X.-Z. Li, Analysis of a SEIV epidemic model with a nonlinear incidence rate. *Appl. Math. Model.* **33** (2009), no. 7, 2919–2926.

[3] V. Chebotaeva and P. A. Vasquez, Erlang-distributed SEIR epidemic models with cross-diffusion. *Mathematics* **11** (2023), 18 pp.

[4] J. P. LaSalle, The stability of dynamical systems. With an appendix: "Limiting equations and stability of nonautonomous ordinary differential equations" by Z. Artstein. In: *Regional Conference Series in Applied Mathematics. Society for Industrial and Applied Mathematics, Philadelphia, PA*, 1976.

[5] M. Y. Li and J. S. Muldowney, A geometric approach to global-stability problems. *SIAM J. Math. Anal.* **27** (1996), no. 4, 1070–1083.

[6] J. Liu and T. Zhang, Global stability for a tuberculosis model. *Math. Comput. Modelling* **54** (2011), no. 1-2, 836–845.

[7] G. Lu and Z. Lu, Geometric approach to global asymptotic stability for the SEIRS models in epidemiology. *Nonlinear Anal. Real World Appl.* **36** (2017), 20–43.

[8] G. P. Sahu and J. Dhar, Analysis of an SVEIS epidemic model with partial temporary immunity and saturation incidence rate. *Appl. Math. Model.* **36** (2012), no. 3, 908–923.

[9] P. van den Driessche and J. Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. *John A. Jacquez memorial volume. Math. Biosci.* **180** (2002), 29–48.

[10] D. Xiao and S. Ruan, Global analysis of an epidemic model with a nonlinear incidence rate. *Math. Biosci.*, 2011 (to appear).