# On Attainability on the Potential from the Space $W_2^{-1}[0,1]$ of the Lower Bound of the First Eigenvalue of a Sturm–Liouville Problem

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Consider the Sturm–Liouville problem

$$y'' + Q(x)y + \lambda y = 0, \quad x \in (0,1), \tag{1}$$

$$y(0) = y(1) = 0, \tag{2}$$

where $Q$ belongs to the set $T_{\alpha,\beta,\gamma}$ of all locally integrable on $(0,1)$ functions with non-negative values such that the following integral conditions hold:

$$\int\limits_0^1 x^\alpha (1-x)^\beta Q^\gamma(x)\,dx = 1, \quad \gamma \neq 0, \tag{3}$$

$$\int\limits_0^1 x(1-x)Q(x)\,dx < \infty. \tag{4}$$

A function $y$ is a *solution* of problem $(1),(2)$ if it is absolutely continuous on the segment $[0,1]$, satisfies (2), its derivative $y'$ is absolutely continuous on any segment $[\rho, 1-\rho]$, where $0<\rho<\frac{1}{2}$, and equality (1) holds almost everywhere in the interval $(0,1)$.

It was proved that if condition (4) does not hold, then for any $0 \le p \le \infty$, there is no non-trivial solution $y$ of equation (1) with properties $y(0)=0$, $y'(0)=p$ ( [4, Theorem 1].

If $\gamma<0$, $\alpha \le 2\gamma-1$ or $\beta \le 2\gamma-1$, then the set $T_{\alpha,\beta,\gamma}$ is empty; for other values $\alpha,\beta,\gamma$, $\gamma \neq 0$, the set $T_{\alpha,\beta,\gamma}$ is not empty [7, Chapter 1, § 2, Theorem 3]. Since for $\gamma<0$, $\alpha \le 2\gamma-1$ or $\beta \le 2\gamma-1$ there exists no function $Q$ satisfying (3) and (4) taken together, we do not consider the problem for these parameters.

Consider the functional

$$R[Q,y] = \frac{\int\limits_0^1 y'^2\,dx - \int\limits_0^1 Q(x)y^2\,dx}{\int\limits_0^1 y^2\,dx}\,.$$

If condition (4) is satisfied, then the functional $R[Q,y]$ is bounded below in $H_0^1(0,1)$ [5]. It was proved [4,5] that for any $Q \in T_{\alpha,\beta,\gamma}$,

$$\lambda_1(Q) = \inf_{y \in H_0^1(0,1)\setminus\{0\}} R[Q,y].$$

In this paper we describe estimates for

$$m_{\alpha,\beta,\gamma} = \inf_{Q \in T_{\alpha,\beta,\gamma}} \lambda_1(Q)$$

for some values of parameters $\alpha$, $\beta$, $\gamma$. The result of the paper is a generalization of a result obtained by one of the authors in [2, 3]. In order to implement the ideas, used in this paper, the authors follow the technique applied in [6] where a similar problem was considered.

Let $\gamma = 1$, $0 \leqslant \alpha, \beta < 1$. For any $Q \in T_{\alpha,\beta,\gamma}$, we have

$$\int_0^1 Q(x) y^2\,dx \leqslant \sup_{[0,1]} \frac{y^2}{x^\alpha (1-x)^\beta} \int_0^1 Q(x) x^\alpha (1-x)^\beta\,dx$$

$$\leqslant \sup_{[0,1]} x^{1-\alpha}(1-x)^{1-\beta} \sup_{[0,1]} \frac{y^2}{x(1-x)} \leqslant \frac{(1-\alpha)^{1-\alpha}(1-\beta)^{1-\beta}}{(2-\alpha-\beta)^{2-\alpha-\beta}} \int_0^1 y'^2\,dx$$

and

$$m_{\alpha,\beta,\gamma} \geqslant \Big(1 - \frac{(1-\alpha)^{1-\alpha}(1-\beta)^{1-\beta}}{(2-\alpha-\beta)^{2-\alpha-\beta}}\Big) \cdot \pi^2 > 0.$$

If $0 \leqslant \alpha, \beta < 1$, $Q \in T_{\alpha,\beta,\gamma}$, then

$$R[Q,y] \geqslant \frac{\int_0^1 y'^2\,dx - \sup\limits_{[0,1]} \frac{y^2}{x^\alpha(1-x)^\beta}}{\int_0^1 y^2\,dx} = L[y].$$

Functional $L$ is bounded below, thus, there exists

$$\inf_{y \in H_0^1(0,1)\setminus\{0\}} L[y] = m.$$

**Theorem.** *If* $0 \leqslant \alpha, \beta < 1$, *then for a point* $x_0 \in (0,1)$ *and a number* $K = {x_0}^{-\alpha}(1-x_0)^{-\beta}$ *we have*

$$m_{\alpha,\beta,1} = m,$$

*where* $m$ *is a solution of the equation*

$$\tan\sqrt{m}\,(1-x_0) = \frac{\sqrt{m}}{K \sin\sqrt{m}\,x_0 - \sqrt{m}\cos\sqrt{m}\,x_0}\,,$$

*and* $m_{\alpha,\beta,1}$ *is attained on the potential* $K\delta(x-x_0)$.

*Proof.* Following [6], we consider $W_2^{-1}[0,1]$, the Hilbert space that is a completion of $L_2[0,1]$ in the norm

$$\|y\|_{W_2^{-1}[0,1]} \rightleftharpoons \sup_{\|z\|_{W_2^1[0,1]}=1} \int_0^1 yz\,dx.$$

For $y \in W_2^{-1}[0,1]$, we denote by $\int_0^1 yz\,dx$ the result

$$\langle y, z\rangle \rightleftharpoons \lim_{n\to\infty} \int_0^1 y_n z\,dx \quad \Big(\text{where } y = \lim_{n\to\infty} y_n, \;\; y_n \in L_2[0,1]\Big)$$

of applying the linear functional $y$ to the function $z \in W_2^1[0,1]$. According to [6], for any function $Q \in T_{\alpha,\beta,\gamma}$ and for any $\lambda \in \mathbb{R}$ we consider the map

$$\begin{gathered} M : W_2^1[0,1] \to L_{loc}[0,1], \\ y \mapsto y'' + (Q+\lambda)y \end{gathered}$$

that for $Q \in W_2^{-1}[0,1]$ can be extended to the operator

$$\begin{gathered} T_Q(\lambda) : W_2^1[0,1] \to W_2^{-1}[0,1], \\ y \mapsto y'' + (Q+\lambda)y. \end{gathered}$$

The result of applying this operator $T_Q(\lambda)$ to a function $z \in W_2^1[0,1]$ is

$$\langle T_q(\lambda)y, z\rangle = \int_0^1 [-y'z' + \lambda yz]\,dx + \langle Qy, z\rangle = \int_0^1 [-y'z' + \lambda yz]\,dx + \lim_{n\to\infty} \int_0^1 (Qy)_n z\,dx,$$

where $\{Qy\}_n$ is a sequence of functions from $L_2[0,1]$.

Let

$$m = \inf_{H_0^1(0,1)\setminus\{0\}} \frac{\int_0^1 y'^2\,dx - \sup\limits_{[0,1]} \frac{y^2}{x^\alpha(1-x)^\beta}}{\int_0^1 y^2\,dx} = L[u],$$

where $u \in H_0^1(0,1)$ is the minimizer of $L$, $x_0$ is one of the points such that

$$\sup_{[0,1]} \frac{u^2}{x^\alpha(1-x)^\beta} = \frac{u^2(x_0)}{x_0^\alpha(1-x_0)^\beta}\,.$$

Denote

$$K = \frac{1}{{x_0}^\alpha(1-x_0)^\beta}\,.$$

Consider the equation

$$y'' + K\delta(x-x_0)y + my = 0 \tag{5}$$

and the boundary conditions

$$y(0) = y(1) = 0. \tag{6}$$

Let us consider the equivalent to (5), (6) boundary value problem

$$y'' + my = 0, \quad (0,x_0)\cup(x_0,1), \tag{7}$$

$$y'(x_0+0) - y'(x_0-0) = -Ky(x_0), \tag{8}$$

$$y(0) = y(1) = 0. \tag{9}$$

Since $y'' = \{y''\} + [y']_{x_0}\delta(x-x_0)$ and $-K\delta(x-x_0)y = -Ky(x_0)$, we have

$$-K\delta(x-x_0)y - my = -my + (y'(x_0+\varepsilon) - y'(x_0-\varepsilon))\delta(x-x_0)$$

and

$$y'(x_0+\varepsilon) - y'(x_0-\varepsilon) = -Ky(x_0).$$

On $[0,x_0)$ we have

$$y = C\sin\sqrt{m}\,x, \quad y' = C\sqrt{m}\,\cos\sqrt{m}\,x,$$
$$y'(x_0-0) = C\sqrt{m}\,\cos\sqrt{m}\,x_0.$$

On $(x_0,1]$ we have

$$y = C\sin\sqrt{m}\,x, \quad y' = C\sqrt{m}\,\cos\sqrt{m}\,x,$$
$$y'(x_0-0) = C\sqrt{m}\,\cos\sqrt{m}\,x_0,$$
$$y = D_1\cos\sqrt{m}\,x + D_2\sin\sqrt{m}\,x,$$
$$D_1 = -D_2\tan\sqrt{m}\,,$$
$$y = -D_2\tan\sqrt{m}\,\cos\sqrt{m}\,x + D_2\sin\sqrt{m}\,x = -D_2\,\frac{\sin\sqrt{m}\,(1-x)}{\cos\sqrt{m}}\,,$$
$$y' = \frac{D_2}{\cos\sqrt{m}}\,\sqrt{m}\,\cos\sqrt{m}\,(1-x),$$
$$y'(x_0+0) = \frac{D_2\sqrt{m}}{\cos\sqrt{m}}\cos\sqrt{m}\,(1-x_0).$$

By virtue of (9), we have

$$\frac{D_2\sqrt{m}}{\cos\sqrt{m}}\,\cos\sqrt{m}\,(1-x_0) - C\sqrt{m}\,\cos\sqrt{m}\,x_0 = -KC\sin\sqrt{m}\,x_0,$$
$$C = \frac{D_2\sqrt{m}\,\cos\sqrt{m}\,(1-x_0)}{\cos\sqrt{m}\,\big(\sqrt{m}\,\cos\sqrt{m}\,x_0 - K\sin\sqrt{m}\,x_0\big)}\,,$$

and

$$y = \begin{cases} \dfrac{D_2\sqrt{m}\,\cos\sqrt{m}\,(1-x_0)}{\cos\sqrt{m}\,\big(\sqrt{m}\cos\sqrt{m}\,x_0 - K\sin\sqrt{m}\,x_0\big)}\sin\sqrt{m}\,x, & x\in[0,x_0], \\ \dfrac{-D_2\sin\sqrt{m}\,(1-x)}{\cos\sqrt{m}}\,, & x\in(x_0,1]. \end{cases}$$

Since $y$ is continuous at $x_0$, we have

$$\frac{\sqrt{m}\,\cos\sqrt{m}\,(1-x_0)}{\sqrt{m}\,\cos\sqrt{m}\,x_0 - K\sin\sqrt{m}\,x_0}\,\sin\sqrt{m}\,x_0 = -\sin\sqrt{m}\,(1-x_0)$$

or

$$\tan\sqrt{m}\,(1-x_0) = \frac{\sqrt{m}}{K\sin\sqrt{m}\,x_0 - \sqrt{m}\,\cos\sqrt{m}\,x_0}\,.$$

In particular [2,3], for $\alpha=\beta=0$, $K=1$, $x_0=\frac{1}{2}$, $m$ is the solution of the equation

$$\tan\frac{\sqrt{m}}{2} = 2\sqrt{m}\,,$$

attained on the potential $\delta(x-\frac{1}{2})$,

$$y = \begin{cases} C\sin\sqrt{m}\,x, & x\in\Big[0,\dfrac{1}{2}\Big], \\ C\sin\sqrt{m}\,(1-x), & x\in\Big(\dfrac{1}{2}\,,1\Big], \end{cases}$$

where $C$ is a constant.

By virtue of

$$\langle T_q(\lambda)y,z\rangle=\int\limits_0^1\big[-y'z'+\lambda yz\big]\,dx+\langle Qy,z\rangle=\int\limits_0^1\big[-y'z'+\lambda yz\big]\,dx+\lim_{n\to\infty}\int\limits_0^1(Qy)_n z\,dx,$$

we have

$$\langle T_q(\lambda)y,y\rangle=\int\limits_0^1\big[-y'^2+my^2\big]\,dx+\langle Qy,y\rangle=\int\limits_0^1\big[-y'^2+my^2\big]\,dx+K{y_0}^2,$$

because if we consider the sequence

$$Q_n(x)=\begin{cases}K\cdot n, & x\in\Big[x_0-\dfrac{1}{2n},x_0+\dfrac{1}{2n}\Big],\\ 0, & x\in\Big[0,x_0-\dfrac{1}{2n}\Big)\cup\Big(x_0+\dfrac{1}{2n},1\Big]\end{cases}$$

and the sequence $\{Qy\}_n$ of functions belonging to $L_2[0,1]$ such that $(Qy)_n=Q_ny$, then

$$\langle Qy,y\rangle=\lim_{n\to\infty}\int\limits_0^1(Qy)_n y\,dx=\lim_{n\to\infty}\int\limits_0^1 Q_ny^2\,dx=K\cdot y^2(x_0).$$

Note that by the mean-value theorem, for any fixed $n$ there exists $x_*\in(x_0-\frac{1}{2n},x_0+\frac{1}{2n})$ such that

$$\frac{1}{2n}\int\limits_{x_0-\frac{1}{2n}}^{x_0+\frac{1}{2n}}K\cdot y^2\,dx=K\cdot\frac{1}{2n}\cdot 2n\cdot y^2(x_*)=K\cdot y^2(x_*).$$

If

$$\langle T_Q(\lambda)y,y\rangle=\int\limits_0^1\big[-y'^2+my^2\big]\,dx+\langle Qy,y\rangle=0,$$

then

$$\int\limits_0^1\big[-y'^2+my^2\big]\,dx+K{y_0}^2=0$$

or

$$\frac{\int\limits_0^1 y'^2\,dx-K{y_0}^2}{\int\limits_0^1 y^2\,dx}=m.$$

Therefore, for the found weak solution $y$ of equation (5), we have

$$m=\frac{\int\limits_0^1 y'^2\,dx-K{y_0}^2}{\int\limits_0^1 y^2\,dx}\geqslant\frac{\int\limits_0^1 u'^2\,dx-K{u_0}^2}{\int\limits_0^1 u^2\,dx}=m,$$

and the weak solution of equation (5) is the minimizer of the functional $L$. □

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