Asymptotic Representation for Solutions of Systems of Differential Equations with Rapidly Varying Nonlinearities

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We consider the system of differential equations

$$\begin{cases} y_1' = \alpha_1 p_1(t)\varphi_2(y_2), \\ y_2' = \alpha_2 p_2(t)\varphi_1(y_1), \end{cases} \tag{1}$$

where $\alpha_i \in \{-1, 1\}$ $(i = 1, 2)$, $p_i : [a, \omega[\, \to]0, +\infty[$ $(i = 1, 2)$ are continuous functions, $-\infty < a < \omega \le +\infty$, $\varphi_i : \Delta(Y_i^0) \to]0; +\infty[$ $(i = 1, 2)$ ($\Delta(Y_i^0)$ is a one-sided neighborhood of Y_i^0, Y_i^0 equals either 0, or $\pm\infty$) are twice continuously differentiable functions that satisfy the conditions

$$\varphi_i'(z) \neq 0 \ \text{ when } \ z \in \Delta(Y_i^0), \qquad \lim_{\substack{z \to Y_i \\ z \in \Delta(Y_i^0)}} \varphi_i(z) = \Phi_i^0 \in \{0, +\infty\},$$

$$\lim_{\substack{z \to Y_i \\ z \in \Delta(Y_i^0)}} \frac{\varphi_i''(z)\varphi_i(z)}{[\varphi_i'(z)]^2} = \gamma_i \ \ (i = 1, 2).$$

Such system of differential equations when $\varphi_i(y_i) = |y_i|^{\sigma_i}$ $(i = \overline{1, n})$ is called the system of differential equations of Emden–Fowler type. While $t \uparrow \omega$, the asymptotic representations for its non-oscillating solutions were established in [2,6]. When $\gamma_i \neq 1$ $(i = 1, 2)$, system (1) is the system with regularly warying nonlinearities. Such system of differential equations had been investigated in [4].

This work considers situation, when $\gamma_1 = 1$, that means function φ_1 is rapidly warying when $y_1 \to Y_1^0$ [1,5]. In this situation, special case of system (1) is a two-term non-autonomous differential equation with rapidly warying nonlinearity (see [3]).

A solution $(y_i)_{i=1}^2$ of system (1), defined on the interval $[t_0, \omega[\, \subset [a, \omega[$, is called $\mathcal{P}_\omega(\Lambda_1, \Lambda_2)$-solution, if functions $u_i(t) = \varphi_i(y_i(t))$ $(i = 1, 2)$ satisfy the following conditions:

$$\lim_{t \uparrow \omega} u_i(t) = \Phi_i^0, \quad \lim_{t \uparrow \omega} \frac{u_i(t)u_{i+1}'(t)}{u_i'(t)u_{i+1}(t)} = \Lambda_i \ \ (i = 1, 2).$$

Note that the second condition in the definition of $\mathcal{P}_\omega(\Lambda_1, \Lambda_2)$-solution implies

$$\prod_{i=1}^{2} L_i = 1.$$

For system (1) in case, when $\Lambda_i \neq 0$ $(i = 1, 2)$, the necessary and sufficient conditions for the existence of $\mathcal{P}_\omega(\Lambda_1, \Lambda_2)$-solutions are established, as well as the asymptotic representation for these solutions when $t \uparrow \omega$.

In order to formulate the theorem, we introduce several auxiliary notations:

$$I_i(t)=\begin{cases}\displaystyle\int\limits_{A_1}^{t} p_1(\tau)\,d\tau & \text{for } i=1,\\ \displaystyle\int\limits_{A_2}^{t} I_1(\tau)p_2(\tau)\,d\tau & \text{for } i=2,\end{cases}\qquad \beta_i=\begin{cases}-\Lambda_1, & \text{if } i=1,\\ -1, & \text{if } i=2,\end{cases}$$

where limits of integration $A_i\in\{\omega,a\}$ are chosen in such a way that corresponding integral I_i aims either to zero, or to ∞ when $t\uparrow\omega$.

$$A_i^*=\begin{cases}1, & \text{if } A_i=a,\\ -1, & \text{if } A_i=\omega\end{cases}\qquad (i=1,2).$$

Theorem. *Let $\Lambda_i\in\mathbb{R}\setminus\{0\}$ $(i=1,2)$ and $\gamma_1=1$. Then for the existence of $\mathcal{P}_\omega(\Lambda_1,\Lambda_2)$ – solutions of* (1) *it is necessary and, if algebraic equation*

$$\nu\big[\nu+(1-\gamma_2)\Lambda_1\big]=1$$

does not have roots with zero real part, it is also sufficient that for each $i=1,2$

$$\lim_{t\uparrow\omega}\frac{I_i(t)I'_{i+1}(t)}{I'_i(t)I_{i+1}(t)}=\Lambda_i\,\frac{\beta_{i+1}}{\beta_i}$$

and following conditions are satisfied

$$A_i^*\beta_i>0 \ \text{ when } \ \Phi_i^0=+\infty,\qquad A_i^*\beta_i<0 \ \text{ when } \ \Phi_i^0=0,$$
$$\operatorname{sign}\big[\alpha_iA_i^*\beta_i\big]=\operatorname{sign}\varphi'_i(z).$$

Moreover, components of each solution of that type admit the following asymptotic representation when $t\uparrow\omega$

$$\frac{\varphi_i(y_i(t))}{\varphi'_i(y_i(t))\varphi_{i+1}(y_{i+1}(t))}=\alpha_i\beta_iI_i(t)[1+o(1)], \quad \text{if } i=1,$$
$$\frac{\varphi_i(y_i(t))}{\varphi'_i(y_i(t))\varphi_{i+1}(y_{i+1}(t))}=\alpha_i\beta_i\,\frac{I_i(t)}{I_1(t)}\,[1+o(1)], \quad \text{if } i=2.$$

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