Boundary Value Problems on the Half-Line Involving Generalized Curvature Operators

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1 Introduction

Consider the second order differential equation

$$\big(a(t)\Phi_C(x')\big)' + b(t)F(x) = 0, \;\; t \in I = [t_0, \infty), \tag{1.1}$$

where the functions a, b are continuous and positive on $I = [t_0, \infty)$, $t_0 \geq 0$, the function F is a continuous function on $\mathbb{R}$ such that $uF(u) > 0$ for $u \neq 0$, $\Phi_R : (-1, 1) \to \mathbb{R}$ and $\Phi_C : \mathbb{R} \to (-1, 1)$ is the monotone homeomorphismus

$$\Phi_C(u) = \frac{|u|^{p-2}u}{(1 + |u|^p)^{(p-1)/p}}\,, \;\; p > 1.$$

The operator Φ_C is called *generalized Euclidean mean curvature operator*. In [4], qualitative similarities between the linear equation

$$(a(t)y')' + b(t)y = 0 \tag{1.2}$$

and equations

$$\big(a(t)\Phi_E(x')\big)' + b(t)F(x) = 0 \;\text{ and }\; \big(a(t)\Phi_M(x')\big)' + b(t)F(x) = 0,$$

are pointed out, where

$$\Phi_E(u) = \frac{u}{\sqrt{1 + |u|^2}} \;\text{ and }\; \Phi_M(u) = \frac{u}{\sqrt{1 - |u|^2}}\,.$$

Operator Φ_C is called *Euclidean mean curvature operator* and Φ_M *Minkowski mean curvature operator*. Operator Φ_E is a special case of Φ_C. Similarly, Φ_M is a particular case of the so called *generalized relativistic operator*

$$\Phi_R(u) = \frac{|u|^{p-2}u}{(1 - |u|^p)^{(p-1)/p}}\,, \;\; p > 1.$$

Curvature operators arise in studying some nonlinear fluid mechanics problems, in particular capillarity-type phenomena for compressible and incompressible fluids, as well as in the relativity theory when some extrinsic properties of the mean curvature of hypersurfaces are considered, see [1, 5, 8–10] and the references therein. In particular, in [10], it is observed that, as for small values of the variable the classical acceleration operator is an approximation of Φ_M and Φ_E; similarly, the p-Laplacian operator Φ_p

$$\Phi_p(u) = |u|^{p-2}u, \quad p > 1, \tag{1.3}$$

can be viewed, again for small values of the variable, as an approximation of Φ_R and Φ_C. Under suitable assumptions on the forcing term F, this similarity between the equation

$$\big(a(t)\Phi_R(x')\big)' + b(t)F(x) = 0 \tag{1.4}$$

and

$$\big(a(t)\Phi_p(x')\big)' + b(t)F(x) = 0$$

is highlighted in the search of periodic solutions, see [10], as well as in other different contexts, concerning the oscillation or the nonoscillation, see [2, Theorem 2.1] and [5, Section 5]. Moreover, in [5] also the existence of solutions x of (1.4) such that $x(t)x'(t) < 0$ on the whole interval I, is considered, jointly with their convergence to zero as $t \to \infty$. These solutions are usually called *global Kneser solutions*. Moreover, their existence and asymptotic behavior have been investigated by many authors for a large variety of equations, see, e.g. [11] and the references therein.

Our aim here is to complete the results in [5, Theorem 4.1], by studying the existence of *global Kneser solution*s for (1.1). Further, also the decay of these solutions near infinity is examined. These results illustrate also that an asymptotic proximity between equations with generalized mean curvature operators and with the p-Laplacian continues to hold for Kneser solutions.

2 A fixed point result

The existence of global Kneser solutions to (1.1) is based on a fixed point result which originates from [3]. It concerns operators $\mathcal{T}$, which are defined in a Fréchet space by a Schauder's quasilinearization device. Roughly speaking, this method reduces the solvability of the given problem to the one of a possibly nonlinear problem, whose solutions have known properties. In particular, this approach does not require the knowledge of the explicit form of the fixed point operator. Moreover, it seems particularly useful when the problem is considered in a noncompact interval. In this case, it permits us also to overcome difficulties, which may originate from the check of topological properties of the fixed point operator, like the compactness, because they become a direct consequence of suitable *a-priori* bounds.

More precisely, we start by reducing the problem to an abstract fixed point equation $x = \mathcal{T}(x)$, where $\mathcal{T}$ is a possible nonlinear operator, defined in a subset of a suitable Fréchet space X. In this approach, an important tool is played by a nice property that connects the operators Φ_C and Φ_R. Indeed, when $p = 2$, the inverse of Φ_E is Φ_M and vice-versa. When $p \neq 2$, denote by q the conjugate number of p, that is

$$q = \frac{p}{p-1}\,. \tag{2.1}$$

Thus a standard calculation shows that if

$$v = \Phi_C(u) = \frac{\Phi_p(u)}{(1 + |u|^p)^{(p-1)/p}}\,, \tag{2.2}$$

then the inverse Φ_C^* of Φ_C is given by

$$u = \Phi_C^*(v) = \frac{\Phi_q(v)}{(1-|v|^q)^{(q-1)/q}}\,, \tag{2.3}$$

that is Φ_C^* reads as Φ_R where p is replaced by q. Indeed, from (2.2) we get

$$|v|^{p/p-1} = |u|^p\big(1+|u|^p\big)^{-1} \ \text{ or } \ 1-|v|^{p/p-1} = \big(1+|u|^p\big)^{-1}.$$

Thus

$$|u| = \frac{|v|^{1/(p-1)}}{(1-|v|^{p/(p-1)})^{1/p}}$$

and so from (2.1) the equality (2.3) follows. In a similar way, the inverse Φ_R^* of Φ_R is given by

$$\Phi_R^*(v) = \frac{\Phi_q(v)}{(1+|v|^q)^{(q-1)/q}}.$$

Using these properties, we can define the fixed point operator in the following way. Let $G : I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be the continuous function such that

$$G(t,\mu,\mu) = b(t)F(\mu) \ \text{ for any } \ (t,\mu)\in I\times\mathbb{R}, \tag{2.4}$$

that is F is the restriction to the diagonal of G. Setting $y = a(t)\Phi_C(x')$, equation (1.1) can be rewritten as the system

$$x' = \Phi_C^*\Big(\frac{y}{a}\Big), \quad y' = -bF(x), \tag{2.5}$$

where, for sake of simplicity, the dependence on the variable t is omitted. Using (2.3), the system (2.5) becomes

$$x' = \big(a^q - |y|^q\big)^{-(q-1)/q}\Phi_q(y), \quad y' = -bF(x). \tag{2.6}$$

Jointly with (2.6), consider the system

$$\xi' = \big(a^q - |v|^q\big)^{-(q-1)/q}\Phi_q(\eta), \quad \eta' = -G(t,u,\xi), \tag{2.7}$$

where the couple (u,v) belongs to a suitable set $\Omega\subset C(I,\mathbb{R}^2)$. If for any $(u,v)\in\Omega$, the system (2.7) has a unique solution (ξ_{uv},η_{uv}) which belongs to a subset $S\subset C(I,\mathbb{R}^2)$, defining $\mathcal{T}(u,v) = (\xi_{uv},\eta_{uv})$ and the operator $\mathcal{T}$ has a fixed point in Ω, then it is easy to verify that the fixed point $(\widehat{x},\widehat{y})$ of $\mathcal{T}$, if any, is a solution of (2.5). In other words, the algebraic aspect of the approach, consists in reducing our problem to one, whose solvability may be more easy. A special case, in which this fact occurs, is when the function F satisfies

$$\lim_{u\to 0}\frac{F(u)}{\Phi_p(u)} = F_0, \ \ 0\le F_0<\infty. \tag{2.8}$$

Indeed, by choosing as G the function $G(t,u,x) = b(t)\widetilde{F}(u(t))\Phi_p(x)$, where

$$\widetilde{F}(u) = \frac{F(u)}{\Phi_p(u)} \ \text{ if } \ u\neq 0, \ \text{ and } \ \widetilde{F}(0) = F_0, \tag{2.9}$$

a standard calculation shows that the system (2.7) is equivalent to the half-linear equation

$$\big(A_v(t)\Phi_p(\xi')\big)' + b(t)\widetilde{F}(u(t))\Phi_p(\xi) = 0, \tag{2.10}$$

where

$$A_v(t) = \Big(a^{p/(p-1)}(t) - |v|^{p/(p-1)}(t)\Big)^{(p-1)/p}. \tag{2.11}$$

For obtaining a fixed point of $\mathcal{T}$, we use the quoted result in [3, Theorem 1.1] and the Tychonoff fixed point theorem. The following holds.

Theorem 2.1. *Let S be a nonempty subset of the Fréchet space $C(I,\mathbb{R}^2)$. Assume that there exists a nonempty, closed, convex and bounded subset $\Omega \subset C(I,\mathbb{R}^2)$ such that, for any $(u,v)\in\Omega$, the system* (2.7) *has a unique solution $(\xi_{uv},\eta_{uv})\in S$. Let $\mathcal{T}$ be the operator $\Omega\to S$, given by $\mathcal{T}(u,v)=(\xi_{uv},\eta_{uv})$. Assume that*

(i_1) *$\mathcal{T}(\Omega)\subset\Omega$;*

(i_2) *if $\{(u_n,v_n)\}\subset\Omega$ is a sequence converging in Ω and $\mathcal{T}((u_n,v_n))\to(\xi_1,\eta_1)$, then $(\xi_1,\eta_1)\in S$.*

Then $\mathcal{T}$ has a fixed point $(\widehat{x},\widehat{y})\in\Omega\cap S$ and $\widehat{x}$ is a solution of (1.1).

An abstract fixed point theorem for equations involving a more general operator is given in [5, Theorem 2.1]. The assumption (i_2) is needed for proving the continuity of $\mathcal{T}$. Indeed, as spite of the fact that in many cases $\mathcal{T}$ turns out to be discontinuous, condition (i_1) becomes necessary and sufficient for the continuity $\mathcal{T}$ when $\mathcal{T}(\Omega)$ is bounded, see [3]. Moreover, condition (i_1) is verified if there exists a closed subset $S_1\subset S\cap\Omega$ such that for any $(u,v)\in\Omega$ the system (2.7) has a unique solution $(\xi_{uv},\eta_{uv})\in S_1$. As claimed, this fact illustrates how the compacteness of $\mathcal{T}$ can be a direct consequence of *a-priori* bounds.

3 Kneser solutions

Here we prove the existence of global Kneser solutions to (1.1), which converge to zero as $t\to\infty$.

Let q be defined by (2.1) and $\Phi_q(u)=|u|^{q-2}u$ be q-Laplacian operator. We assume (2.8),

$$J_a=\int_{t_0}^{\infty}\Phi_q(a^{-1}(s))\,ds<\infty,\quad \inf_{t\ge t_0}\Phi_q(a(t))\int_{t}^{\infty}\Phi_q(a^{-1}(s))\,ds=\lambda>0, \tag{3.1}$$

and

$$\int_{t_0}^{\infty}\Phi_q\Big(a^{-1}(t)\int_{t_0}^{t}b(s)\,ds\Big)\,dt<\infty\,,\quad \int_{t_0}^{\infty}b(t)\Phi_p\Big(\int_{t}^{\infty}\Phi_q(a^{-1}(s))\,ds\Big)\,dt<\infty. \tag{3.2}$$

Choose $0<c<\lambda$ and set

$$K=(1-c\lambda^{-1})^{(p-1)/p},\quad M_F=\max_{u\in[0,c]}\widetilde{F}(u), \tag{3.3}$$

where $\widetilde{F}$ is given in (2.9). Consider the half-linear equation

$$\big(Ka(t)\,\Phi_p(z')\big)'+M_Fb(t)\Phi_p(z)=0. \tag{3.4}$$

The following holds.

Theorem 3.1. *Let* (2.8), (3.1) *and* (3.2) *be satisfied. If* (3.4) *is nonoscillatory and its principal solution z_0 is positive decreasing on I, then* (1.1) *has infinitely many global Kneser solutions, which converge to zero as $t\to\infty$.*

The proof of Theorem 3.1 is similar to the one given in [5, Theorem 4.1] for proving the existence of global Kneser solutions of (1.4), with some modifications. It is based on Theorem 2.1 and on some comparison properties between principal solutions of half-linear equations. We start by recalling these properties. The notion of principal solution, introduced in 1936 by Leighton & Morse for the linear equation (1.2), has been extended to the half-linear equation

$$\big(a(t)\,\Phi_p(x')\big)'+b(t)\Phi_p(x)=0 \tag{3.5}$$

by Elbert & Kusano and independently by Mirzov, using the associated generalized Riccati equation, see [7] for more datails. More precisely, if (3.5) is nonoscillatory, then a nontrivial solution x_0 of (3.5) is said to be *the principal solution* if for every nontrivial solution x of (3.5) such that $x \neq \mu x_0$, $\mu \in \mathbb{R}$, the inequality

$$\frac{x_0'(t)}{x_0(t)} < \frac{x'(t)}{x(t)} \quad \text{for large } t$$

holds. The set of principal solutions of (3.5) is nonempty and principal solutions are determined up to a constant factor. If x is a solution of (3.5), we denote its *quasiderivative* $x^{[1]}$ by $x^{[1]}(t) = a(t)\,\Phi_\alpha(x'(t))$. The following comparison property plays a crucial role in the proof of Theorem 3.1.

Lemma 3.1. *Assume $J_a < \infty$. If* (3.5) *is nonoscillatory and its principal solution x_0, starting at $x_0(t_0) = k > 0$, is positive decreasing on I, then the principal solution y_0 of any minorant of* (3.5)*, starting at $y(t_0) > 0$, is positive decreasing on the whole interval I and satisfies the inequality*

$$\frac{x_0^{[1]}(t)}{\Phi_p(x_0(t))} < \frac{y_0^{[1]}(t)}{\Phi_p(y_0(t))} \quad \textit{for any } t \in I.$$

Now, we give a sketch of the proof of Theorem 3.1. Set $H = \Phi_q(K)$, where K is given in (3.3) and, without loss of generality, suppose $z_0(t_0) = c^H$. In view of (3.2) we have $\lim_{t\to\infty} z_0(t) = 0$, see, e.g., [5, Proposition 3.2]. Let Ω be the set

$$\Omega = \Big\{(u,v) \in C(I,\mathbb{R}^2):\ \ 0 \le u(t) \le (z_0(t))^H,\ \ u(t_0) = 0,\ \ -\Phi_p(c\lambda^{-1})a(t) \le v(t) \le 0\Big\}.$$

For any $(u,v) \in \Omega$, the half-linear equation (2.10) is a minorant of (3.4). Thus, (2.10) is nonoscillatory and, from Lemma 3.1, its principal solution η_{uv} such that $\eta_{uv}(t_0) = c$, is positive decreasing on I. Moreover, we have for any $t \in I$

$$\frac{\eta_{uv}^{[1]}(t)}{\Phi_p(\eta_{uv}(t))} \le \frac{K a(t)\,\Phi_p(z_0'(t))}{\Phi_p(z_0(t))}\,.$$

From this, using $Ka(t) \le A_v(t) \le a(t)$ and taking into account that $z_0'(t) < 0$, with a standard calculation we get

$$\eta_{uv}(t) \le (z_0(t))^H\,. \tag{3.6}$$

Let w_0 be the principal solution of equation $(a(t)\Phi_p(w'))' = 0$ such that $w_0(t_0) = c$, i.e.,

$$w_0(t) = c\bigg(\int_{t_0}^{\infty} \Phi_q(a^{-1}(s))\,ds\bigg)^{-1} \int_t^{\infty} \Phi_q(a^{-1}(s))\,ds.$$

Again from Lemma 3.1, we obtain

$$\frac{-1}{\Phi_p(w_0(t))} = \frac{w_0^{[1]}(t)}{\Phi_p(w_0(t))} \le \frac{\eta_{uv}^{[1]}(t)}{\Phi_p(\eta_{uv}(t))}\,.$$

From this, since $\eta_{uv}(t) \le \eta_{uv}(t_0) = c$, we have

$$\begin{aligned}|\eta_{uv}^{[1]}(t)| &\le \Phi_p\bigg(c\bigg(\int_t^{\infty} \Phi_q(a^{-1}(s))\,ds\bigg)^{-1}\bigg)\\ &= a(t)\Phi_p\bigg(c\bigg(\Phi_q(a(t)\int_t^{\infty} \Phi_q(a^{-1}(s))\,ds\bigg)^{-1}\bigg) \le \Phi_p(c\lambda^{-1})a(t).\end{aligned}$$

From this and (3.6) we have $(\eta_{uv}, \eta_{uv}^{[1]}) \in \Omega$. Using (3.2) and the same argument to the one given in [5, Theorem 4.1], for any $(u, v) \in \Omega$, the couple $(\eta_{uv}, \eta_{uv}^{[1]})$ is the only pair $(\xi, \xi^{[1]})$ with ξ solution of (2.10), that belongs to Ω. Let $\mathcal{T}$ be the operator $\mathcal{T}(u, v) = (\eta_{uv}, \eta_{uv}^{[1]})$. By choosing $S = \Omega$, conditions (i_1) and (i_2) of Theorem 2.1 are verified. Thus, $\mathcal{T}$ has a fixed point, and the assertion follows.

We close the paper with some comments.

(1) Theorem 3.1 requires that (3.5) is nonoscillatory and its principal solution starting at a positive value at t_0, is positive decreasing for any $t \in I$. To check this property, we may use the generalized Euler equations

$$\big(t^p \Phi_p(x')\big)' + p^{-p} \Phi_p(x) = 0, \ \ t \geq t_0 > 0 \tag{3.7}$$

or

$$(t^n \Phi_p(x'))' + \Big(\frac{n-p+1}{p}\Big)^p t^{n-p} \Phi_p(x) = 0, \ \ p > 2, \ \ n > p-1, \ \ t \geq t_0 > 0, \tag{3.8}$$

see [5, Corollary 4.3 and (4.18)] and [6, Corollary 1], respectively. The principal solution of (3.7) is $\varphi(t) = t^{-p}$ and that of (3.8) is $\varphi(t) = t^{-p}$. Thus, using (3.7) and applying Lemma 3.1, equation (3.5) is nonoscillatory and its principal solution starting at a positive value at t_0, is positive decreasing for any $t \in I$, if for $t \geq t_0$

$$Ka(t) \geq t^p \ \text{ and } \ M_F b(t) \leq p^{-p}.$$

Clearly, a similar result can be formulated by using (3.8).

(2) The proof of Theorem 3.1 yields also the rate of the decay to zero for global Kneser solutions of (1.1). Indeed, using (3.2) and [5, Proposition 3.2], for any $(u, v) \in \Omega$, the principal solution $\eta_{uv}^{[1]}$ satisfies $\lim\limits_{t\to\infty} |\eta_{uv}^{[1]}(t)| = \ell_\eta$, $0 < \ell_\eta < \infty$. From this, it is easy to obtain

$$\eta_{uv}(t) = O\bigg(\int\limits_t^\infty \Phi_q(a^{-1}(s))\,ds\bigg) \ \text{ for large } t.$$

(3) Another interesting case in which Theorem 2.1 can be applied is when the function G in (2.4) is

$$G(t, u, x) = b(t)\widetilde{F}(u(t))\Phi_r(x), \ \ r \neq p.$$

In this case the system (2.7) becomes equivalent to the generalized Emden–Fowler equation

$$\big(a(t)\Phi_p(x')\big)' + b(t)F(x) = 0.$$

Thus, the asymptotic behavior of solutions of (1.1) can be examined via properties of solutions of a suitable Emden–Fowler type equation. This will be done in a forthcoming paper.

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