Adomian Decomposition Method in Theory of Nonlinear Periodic Boundary Value Problems

Sergey Chuiko[1,2]

[1] *Donbas State Pedagogic University, Donetsk Region, Slovyansk, Ukraine*
[2] *Max Planck Institute for Dynamics of Complex Technical Systems, Magdeburg, Germany*
E-mail: chujko-slav@ukr.net

Alexey Chuiko, Nikita Popov

Donbass State Pedagogic University, Donetsk Region, Slavyansk, Ukraine

For the nonlinear boundary-value problem for an ordinary differential equation in the critical and noncritical cases, we obtain constructive conditions of its solvability and the scheme for finding solutions by using Adomian decomposition method.

We investigate the problem of construction of solution [3, 6, 7]

$$z(\,\cdot\,,\varepsilon) \in \mathbb{C}^1[0,T], \;\; z(t,\,\cdot\,) \in \mathbb{C}[0,\varepsilon_0]$$

of the nonlinear periodic boundary-value problem

$$\frac{dz}{dt} = A\,z + f(t) + Z(z,t), \;\; \ell z(\,\cdot\,) := z(0) - z(T) = 0 \tag{1}$$

in a small neighborhood of the solution of the generating problem

$$\frac{dz_0}{dt} = A\,z_0, \;\; \ell z_0(\,\cdot\,) := z_0(0) - z_0(T) = 0, \tag{2}$$

where A is a constant $(n \times n)$-dimensional matrix, $Z(z,t)$ is a nonlinear vector function analytic in the unknown z in a small neighborhood of the solution of the generating problem (2). In addition, the vector function $Z(z,t)$ and the function $f(t)$ are continuous in the independent variable t on the segment $[a,b]$.

The urgency of investigation of the boundary-value problem (1) is explained by extensive applications of similar problems in the study of nonisothermal chemical reactions. An example of simulation of these reactions can be found in [2].

At the end of the present paper, we give an example of determination of approximations to a periodic solution of problem (1) obtained by using our iterative scheme. In [4, 5], approximations to the solutions of nonlinear boundary-value problems and, in particular, periodic boundary-value problems, were found by using the effective Newton–Kantorovich method [9].

In constructing solutions of nonlinear boundary-value problems, we encounter the problem of impossibility of representation of these solutions in terms of elementary functions, which, in turn, leads to the appearance of significant errors in the solutions of the analyzed problems. A similar situation was demonstrated for a periodic problem posed for the equation used to describe the motion of a satellite on the elliptic orbit [11].

In addition, the procedure of construction of the solutions of nonlinear boundary-value problems by the method of simple iterations is significantly complicated by the necessity of evaluation of the derivatives of nonlinearities [6]. In [4, 5], the rate of convergence of iterations was improved as a result of evaluation of the derivatives of nonlinearities in each step. In view of these difficulties, we can expect that the procedure of evaluation of the derivatives of nonlinearities can be simplified and the solutions of nonlinear boundary-value problems (and, in particular, of periodic boundary-value problems) can be found in terms of elementary functions by using the Adomian decomposition method [1]. An example of this simplification is presented in [8].

By $X(t)$ we denote a normal ($X(a) = I_n$) fundamental matrix of the generating problem (2). In the critical case, we have

$$\det Q = 0$$

and the generating problem (2) under the condition [6]

$$P_{Q_r^*} \ell K[f(s)](\,\cdot\,) = 0 \tag{3}$$

has an r-parameter family of solutions

$$z_0(t, c_r) = X_r(t)c_r + G[f(s)](t), \ \ c_r \in \mathbb{R}^r.$$

Here, the matrix $X_r(t)$ consists of r-linearly independent columns of the normal fundamental matrix $X(t)$. The matrix $P_{Q_r^*}$ is formed by r linearly independent rows of the matrix orthoprojector. Furthermore,

$$G[g(s)](t) := K[g(s)](t) - X(t)Q^+ \ell K[g(s)](\,\cdot\,)$$

is the generalized Green operator of the periodic boundary-value problem [6]

$$\frac{dy}{dt} = A\,y + g(t), \quad y(0) - y(T) = 0$$

in the critical case and Q^+ is the pseudoinverse Moore-–Penrose matrix. It is known that the critical case occurs if and only if the matrix A has eigenvalues on the imaginary axis, namely, imaginary numbers of the form

$$\lambda = \frac{2\pi ik}{T}\,, \ \ k = 0, 1, 2, \dots, \ \ i = \sqrt{-1}\,.$$

The necessary and sufficient condition for the solvability of problem (1)

$$P_{Q_r^*}\,\ell K\big[Z(z(s), s)\big](\,\cdot\,) = 0$$

leads to a necessary condition for the solvability of problem (2) in a small neighborhood of the solution of the generating T-periodic problem

$$F_0(c_r) := P_{Q_r^*}\,\ell K\big[A_0(z_0(s, c_r), s)\big](\,\cdot\,) = 0. \tag{4}$$

In what follows, equation (4) is called the equation for generating amplitudes of the $T-$periodic problem (1). Assume that the equation for generating amplitudes (4) has real roots. Fixing one of real solutions $c_r^* \in \mathbb{R}^r$ of equation (4), we arrive at the problem of construction of a solution to the nonlinear $T-$periodic problem (1) in a small neighborhood of the solution

$$z_0(t, c_r^*) = X_r(t)c_r^* + G[f(s)](t), \ \ c_r^* \in \mathbb{R}^r$$

of the generating $T-$periodic problem (2). The conventional condition of solvability of problem (1) in a small neighborhood of the generating T-periodic problem (2) is the requirement of simplicity of the roots [6]

$$\det B_0 \neq 0, \quad B_0 := F_0'(c_0) \in \mathbb{R}^{r\times r}$$

of the equation for generating amplitudes (4) of the T-periodic problem (1). The form of the matrix B_0 which plays the key role in the investigation of the T-periodic problem (1) with the use of the Adomian decomposition [1, c. 502], coincides with the conventional form [6]

$$B_0 = P_{Q_r^*}\,\ell K[\mathcal{A}_1(s)X_r(s)](\,\cdot\,), \quad \mathcal{A}_1(t) = \left.\frac{\partial Z(z,t)}{\partial z}\right|_{z=z_0(t,c_r^*)}$$

is an $(n \times n)$-dimensional matrix. We seek the solution of the periodic boundary-value problem (1) in the form

$$z(t) := z_0(t,c_r^*) + u_1(t) + \cdots + u_k(t) + \cdots .$$

Since the nonlinear vector function $Z(z,t)$ is analytic in the unknown z in a neighborhood of the solution $z_0(t,c_r^*)$ of the generating problem (2), the following decomposition is true in this neighborhood [1, p. 502]

$$Z(z(t),t) = A_0(z_0(t,c_r^*),t) + A_1\big(z_0(t,c_r^*),u_1(t),t\big) + \cdots + A_n\big(z_0(t,c_r^*),u_1(t),\ldots,u_n(t),t\big) + \cdots . \quad (5)$$

The first approximation to the solution of the nonlinear periodic boundary-value problem (1) in the critical case

$$z_1(t,c_r^*) := z_0(t,c_r^*) + u_1(t), \quad u_1(t) = X_r(t)c_1 + G\big[A_0(z_0(s,c_r^*))\big](t), \;\; c_1 \in \mathbb{R}^r$$

is given by the solution of the nonlinear periodic boundary-value problem of the first approximation

$$u_1'(t) = A\,u_1(t) + A_0(z_0(t,c_r^*)), \quad u_1(0) - u_1(T) = 0.$$

The periodicity of solution to the boundary-value problem of the first approximation is guaranteed by the choice of the solution $c_r^* \in \mathbb{R}^r$ of equation (4). The second approximation to the solution of the nonlinear periodic boundary-value problem (1) in the critical case

$$z_2(t,c_r^*) := z_0(t,c_r^*) + u_1(t,c_1) + u_2(t,c_2)$$

is given by the solution of the nonlinear periodic boundary-value problem of the second approximation

$$u_2'(t) = A\,u_2(t) + A_1\big(z_0(t,c_r^*),u_1(t,c_1)\big), \quad u_2(0) - u_2(T) = 0,$$

where

$$u_2(t) = X_r(t)c_2 + G\big[A_1\big(z_0(s,c_r^*),u_1(s,c_1)\big)\big](t), \;\; c_2 \in \mathbb{R}^r.$$

The condition of solvability of the boundary-value problem of the second approximation

$$F_1(c_1) := P_{Q_r^*}\,\ell K\big[A_1\big(z_0(s,c_r^*),u_1(s,c_1)\big)\big](\,\cdot\,) = 0$$

is a linear equation

$$F_1(c_1) = B_0\,c_1 + d_1 = 0, \quad (6)$$

which is uniquely solvable in the case where the matrix B_0 is nondegenerate; here,

$$B_0 = F_1'(c_1) \in \mathbb{R}^{r\times r}, \quad d_1 := F_1(c_1) - B_0\,c_1.$$

Indeed, consider a vector function [10]

$$v(t,\varepsilon) := z_0(t,c_r^*) + \varepsilon\,u_1(t,c_1) + \cdots + \varepsilon^k\,u_k(t,c_k) + \cdots .$$

In this case,

$$F_1(c_1) := P_{Q_r^*}\,\ell K\big[A_1\big(z_0(s,c_r^*),u_1(s,c_1)\big)\big](\,\cdot\,) \\ = P_{Q_r^*}\,\ell K\big[Z'_\varepsilon(v(s,\varepsilon),s))\big](\,\cdot\,)\Big|_{\varepsilon=0} = P_{Q_r^*}\,\ell K\big[\mathcal{A}_1(s)u_1(s,c_1)\big](\,\cdot\,).$$

Thus,

$$B_0 = F_1'(c_1).$$

Therefore, under the condition of simplicity of roots of the equation for generating amplitudes (4) of the periodic problem (1), we obtain the following solution of the boundary-value problem of the first approximation:

$$u_1(t) = X_r(t)c_1 + G\big[A_0(z_0(s,c_r^*))\big](t), \quad c_1 = -B_0^{-1}\,d_1.$$

The conditions of solvability of the boundary-value problems in the next approximations have the form of linear equations. A sequence of approximations to the solution on the nonlinear periodic boundary-value problem (1) in the critical case is given by the following iterative scheme:

$$\begin{gathered} z_1(t,c_r^*) := z_0(t,c_r^*) + u_1(t),\ u_1(t) = X_r(t)c_1 + G\big[A_0(z_0(s,c_r^*))\big](t),\ \ c_1 = -B_0^{-1}\,d_1,\dots, \\ z_{k+1}(t,c_r^*) := z_0(t,c_r^*) + u_1(t,c_1) + \cdots + u_{k+1}(t,c_k),\ \ k=0,1,2,\dots\,, \\ u_{k+1}(t) = X_r(t)c_{k+1} + G\big[A_k(z_0(s,c_r^*),u_1(s,c_1),\dots,u_k(s,c_k))\big](t),\ \ c_k = -B_0^{-1}\,d_k. \end{gathered} \tag{7}$$

Theorem. *In the critical case the generating periodic boundary-value problem* (2) *with condition* (3) *has an r-parameter family of solutions*

$$z_0(t,c_r) = X_r(t)c_r + G[f(s)](t), \ \ c_r \in \mathbb{R}^r.$$

Moreover, if the problem of construction of a solution to the nonlinear periodic boundary-value problem (1) *in a small neighborhood of the solution of the generating problem* (2) *is solvable in the critical case, then the equation for generating amplitudes* (4) *of the T-periodic problem* (1) *has real roots. In the case, where the matrix B_0 is nondegenerate, the iterative scheme* (7) *gives a sequence of approximations to the solution of the T-periodic boundary-value problem* (1) *in the critical case. If there exists a constant $0<\gamma<1$, for which the inequality*

$$\|u_1(t,c_1)\|_\infty \le \gamma\|z_0(t)\|_\infty, \quad \|u_{k+1}(t,c_{k+1})\|_\infty \le \gamma\|u_k(t,c_k)\|_\infty,\ \ k=1,2,\dots, \tag{8}$$

is true, then the iterative scheme (7) *converges to the solution of the nonlinear periodic boundary-value problem* (1).

References

[1] G. Adomian, A review of the decomposition method in applied mathematics. *J. Math. Anal. Appl.* **135** (1988), no. 2, 501–544.

[2] P. Benner, A. Seidel-Morgenstern and A. Zuyev, Periodic switching strategies for an isoperimetric control problem with application to nonlinear chemical reactions. *Appl. Math. Model.* **69** (2019), 287–300.

[3] A. A. Boĭchuk, Nonlinear boundary value problems for systems of ordinary differential equations. (Russian) *Ukraïn. Mat. Zh.* **50** (1998), no. 2, 162–171; translation in *Ukrainian Math. J.* **50** (1998), no. 2, 186–195.

[4] A. A. Boĭchuk and S. M. Chuĭko, On the approximate solution of nonlinear boundary value problems by the Newton–Kantorovich method. (Russian) *Nelīnīĭnī Koliv.* **23** (2020), no. 2, 162–183; translation in *J. Math. Sci. (N.Y.)* **258** (2021), no. 5, 594–617.

[5] O. A. Boĭchuk and S. M. Chuĭko, On the approximate solution of weakly nonlinear boundary value problems by the Newton–Kantorovich method. (Ukrainian) *Nelīnīĭnī Koliv.* **23** (2020), no. 3, 321–331; translation in *J. Math. Sci. (N.Y.)* **261** (2022), no. 2, 228–240.

[6] A. A. Boichuk and A. M. Samoilenko, *Generalized Inverse Operators and Fredholm Boundary-Value Problems*. Second edition. Translated from the Russian by Peter V. Malyshev. Inverse and Ill-posed Problems Series, 59. De Gruyter, Berlin, 2016.

[7] S. M. Chuĭko, On the domain of convergence of an iterative procedure for an autonomous boundary value problem. (Russian) *Nelīnīĭnī Koliv.* **9** (2006), no. 3, 416–432; translation in *Nonlinear Oscil. (N.Y.)* **9** (2006), no. 3, 405–422.

[8] S. M. Chuiko, O. S. Chuiko and M. V. Popov, Adomian decomposition method in the theory of nonlinear boundary-value problems. (Ukrainian) *Nelīnīĭnī Koliv.* **25** (2022), no. 4, 41–425; translation in *J. Math. Sci. (N.Y.)* **277** (2023), no. 2, 338–351.

[9] L. W. Kantorowitsch and G. P. Akilow, *Funktionalanalysis in Normierten Räumen.* (German) In deutscher Sprache herausgegeben von P. Heinz Müller. Mathematische Lehrbücher und Monographien, II. Abteilung, Mathematische Monographien, Band XVII. Akademie-Verlag, Berlin, 1964.

[10] M. K. Mak, C. S. Leung and T. Harko, A Brief introduction to the adomian decomposition method, with applications in astronomy and astrophysics. Preprint arXiv:2102.10511; https://arxiv.org/abs/2102.10511

[11] A. M. Samoĭlenko, S. M. Chuĭko and O. V. Nēsmēlova, Nonlinear boundary value problems unsolved with respect to the derivative. (Ukrainian) *Ukraïn. Mat. Zh.* **72** (2020), no. 8, 1106–1118; translation in *Ukrainian Math. J.* **72** (2021), no. 8, 1280–1293.