On the Solvability of Linear Functional Differential Equations of the First and Second Orders

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We consider the functional differential equations, which can be written in the operator form:

$$(\mathcal{L}_1 x)(t) \equiv \dot{x}(t) - (T^+x)(t) + (T^-x)(t) = f(t), \quad t \in [0,1], \tag{1}$$

$$(\mathcal{L}_2 x)(t) \equiv \ddot{x}(t) - (T^+x)(t) + (T^-x)(t) = f(t), \quad t \in [0,1], \tag{2}$$

where T^+ and T^- are linear positive operators acting from the space of real continuous functions $\mathbf{C}[0,1]$ into the space of real integrable functions $\mathbf{L}[0,1]$ (positive operators map non-negative functions into non-negative ones), $f \in \mathbf{L}[0,1]$ is integrable. The solution of equation (1) (equation (2)) is an absolutely continuous function on $[0,1]$ (a function with an absolutely continuous derivative on $[0,1]$) that satisfies the equation for almost all $t \in [0,1]$.

Numerous studies have explored the solvability conditions of a wide range of boundary value problems associated with functional differential equations (1), (2), in particular, the periodic, Cauchy, antiperiodic, and other types of boundary value problems [1–3, 5].

The literature concerning solvability conditions for functional differential equations, as distinct from boundary value problems for these equations, is notably sparse, if not entirely absent. However, the question of the solvability of the functional differential equation itself is nontrivial, since we do not require the operators to be Volterra, hence, in particular, the Cauchy problem may not have a solution. To the best of the author's knowledge, simple coefficient conditions for the solvability of equations (1), (2) have not yet been formulated. Our goal is to fill this gap and obtain unimprovable sufficient conditions for solvability in terms of the norms of the positive operators T^+ and T^-.

Such operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ have the representation [4, p. 317] in the form of the Stieltjes integral

$$(Tx)(t) = \int_0^1 x(s)\, d_s r(t,s), \quad t \in [0,1],$$

where $r(t,\cdot\,) \in \mathbf{L}[0,1]$ is nondeccreasing for almost all $t \in [0,1]$, the function $t \to r(t,1) - r(t,0)$ is integrable on $[0,1]$. We assume that $r(t,0) \equiv 0$ for all $t \in [0,1]$. The norm of such an operator is defined by the equality

$$\|\,T\|_{\mathbf{C}\to\mathbf{L}} = \int_0^1 (T\mathbf{1}\,)(t)\, dt = \int_0^1 r(t,1)\, dt,$$

where $\mathbf{1}$ is the unit function.

Definition. We will call equation (1) or (2) everywhere solvable if for each function $f \in \mathbf{L}[0,1]$ there is at least one solution.

For first order equations, it was relatively easy to show that if the following conditions (4) or (5) are satisfied, then for the equation $\mathcal{L}_1 x = f$ one of the boundary value problems

$$x(0) = x(1), \quad x(0) = 0, \quad x(1) = 0$$

is uniquely solvable. Thus, the equation $\mathcal{L}_1 x = f$ is solvable everywhere. If conditions (4) and (5) are not satisfied, then there exist linear positive operators T^+, T^- : $\mathbf{C}[0,1] \to \mathbf{L}[0,1]$, for which the equalities (3) are satisfied and $\mathcal{L}_1(\mathbf{AC}[0,1]) \neq \mathbf{L}[0,1]$.

Theorem 1. *The equation* (1) *is everywhere solvable for all linear positive operators* T^+, T^- : $\mathbf{C}[0,1] \to \mathbf{L}$ *satisfying equalities*

$$\| T^+ \|_{\mathbf{C}\to\mathbf{L}} = \mathcal{T}^+, \quad \| T^- \|_{\mathbf{C}\to\mathbf{L}} = \mathcal{T}^-, \tag{3}$$

if and only if non-negative numbers $\mathcal{T}^+$, $\mathcal{T}^-$ *satisfy the inequalities*

$$\mathcal{T}^+ < 1, \quad \mathcal{T}^- < 2\big(1 + \sqrt{1 - \mathcal{T}^+}\big) \tag{4}$$

or the inequalities

$$\mathcal{T}^- < 1, \quad \mathcal{T}^+ < 2\big(1 + \sqrt{1 - \mathcal{T}^-}\big). \tag{5}$$

In the study of the equation (2), the adjoint operator

$$\mathcal{L}_2^* : \mathbf{L}_\infty[0,1] \to (\mathbf{AC}^1[0,1])^* \simeq \mathbf{L}_\infty[0,1] \times \mathbb{R}^2$$

is used.

Since $\mathcal{L}_2$ is a Noetherian operator of index 2, the equation $\mathcal{L}_2$ is everywhere solvable if and only if the homogeneous equation with the adjoint operator

$$\mathcal{L}_2^* g = 0_{(\mathbf{AC}^1[0,1])^*}, \quad g \in \mathbf{L}_\infty[0,1], \tag{6}$$

has only the trivial solution.

If the function $g \in \mathbf{L}_\infty[0,1]$ is a solution to the equation (6), then the function g is absolutely continuous and satisfies the following boundary value problem:

$$\dot{g}(t) = \int\limits_0^1 r(s,t) g(s)\, ds, \quad t \in [0,1], \tag{7}$$

$$g(0) = 0, \quad g(1) = 0, \quad \int\limits_0^1 r(s,1) g(s)\, ds = 0, \tag{8}$$

where $r(s,t) = r^+(s,t) - r^-(s,t)$,

$$(T^+ x)(t) = \int\limits_0^1 x(s)\, d_s r^+(t,s), \quad (T^- x)(t) = \int\limits_0^1 x(s)\, d_s r^-(t,s), \quad t \in [0,1].$$

When studying the system (7), (8), we find that if for given $\mathcal{T}^+$, $\mathcal{T}^-$ there exists a nontrivial solution of this system for some operators T^+, T^- satisfying the equalities (3), then the system (7), (8) has a piecewise linear solution (possibly for other operators T^+, T^- satisfying the equalities (3)). Such a solution corresponds to some operators T^+, T^- of the following form:

$$(T^+ x)(t) = \sum_{j=1}^n p_j^+(t) x(t_j), \quad (T^- x)(t) = \sum_{j=1}^n p_j^-(t) x(t_j), \quad t \in [0,1], \tag{9}$$

the integrable functions p_j^+, p_j^- are non-negative,

$$0 \le t_1 < t_2 < \cdots < t_n \le 1.$$

For operators of the form (9), for which the equalities

$$\sum_{j=1}^{n} \| p_j^+ \|_{\mathbf{L}} = \mathcal{T}^+, \quad \sum_{j=1}^{n} \| p_j^- \|_{\mathbf{L}} = \mathcal{T}^-,$$

are satisfied, the solvability conditions for problem (7), (8) are formulated explicitly.

Let's introduce the following notation:

$$G(k) = \left(1 + \sqrt{k} + \sqrt{k+1}\right)^2 (k+1),$$
$$H_1(k) = \frac{G(k) - \mathcal{T}^-}{k}, \quad H_2(k) = G(k) - k\,Q,$$
$$\widetilde{P}_1(\mathcal{T}^-) \equiv \min_{k\in(0,1]} H_1(k, \mathcal{T}^-), \quad \widetilde{P}_2(\mathcal{T}^-) \equiv \min_{k\in[0,1]} H_2(k, \mathcal{T}^-).$$

Remark. Note that $\widetilde{P}_1(\mathcal{T}^-)$ decreases on $[0,4]$ and can be defined parametrically:

$$\mathcal{T}^- = G(k) - \frac{dG(k)}{dk}\,k, \quad \widetilde{P}_1(\mathcal{T}^-) = \frac{dG(k)}{dk}\,, \quad k \in [k_0, 1],$$

where $k_0 = k_1^2 \approx 0.43$, $k_1 \in [0,1]$ is the only root of the equation $k^4 + 6k^3 + 5k^2 - k = 0$ on the interval $[0,1]$.

The function $\widetilde{P}_2$ is equal to 4 on $[4, \widetilde{P}_1(4)]$, where $\widetilde{P}_1(4) \approx 17.7$; on the interval $[\widetilde{P}_1(4), 12+8\sqrt{2}]$ the function $\widetilde{P}_2(\mathcal{T}^-)$ decreases and can also be specified parametrically:

$$\widetilde{P}_2(\mathcal{T}^-) = G(k) - \frac{dG(k)}{dk}\,k, \quad \mathcal{T}^- = \frac{dG(k)}{dk}\,.$$

Theorem 2. *Let non-negative $\mathcal{T}^+$ and $\mathcal{T}^-$ be given. The equation* (2) *is everywhere solvable for all positive linear operators T^+, T^- : $\mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that equalities* (3) *hold, if and only if*

$$\mathcal{T}^- \in [0,4], \quad \mathcal{T}^+ \le \widetilde{P}_1(\mathcal{T}^-),$$

or

$$\mathcal{T}^- \in (4, 12 + 8\sqrt{2}], \quad \mathcal{T}^+ \le \widetilde{P}_2(\mathcal{T}^-).$$

Corollary. *Let non-negative $\mathcal{T}$ be given. Each of the equations*

$$\ddot{x}(t) - (Tx)(t) = f(t), \quad \ddot{x}(t) + (Tx)(t) = f(t), \ \ t \in [0,1],$$

is everywhere solvable for all linear positive operators $T : \mathbf{C}[0,1] \to \mathbf{L}[0,1]$ such that $\| T\|_{\mathbf{C}\to\mathbf{L}} = \mathcal{T}$, if and only if

$$\mathcal{T} \le 12 + 8\sqrt{2}\,.$$

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