Radial Properties of Stability and Instability of a Differential System

A. A. Bondarev, I. N. Sergeev

Lomonosov Moscow State University, Moscow, Russia

E-mails: albondarev1998@yandex.ru; igniserg@gmail.com

For a natural number $n \in \mathbb{N}$ and a domain G of the Euclidean space $\mathbb{R}^n$ with the Lebesgue measure mes, consider a differential system of the form

$$\dot{x} = f(t,x), \quad x \in G, \quad f(t,0) = 0, \quad t \in \mathbb{R}_+ \equiv [0,+\infty), \quad f, f'_x \in C(\mathbb{R}_+ \times G). \tag{1}$$

Let $B_\delta \equiv \{x_0 \in \mathbb{R}^n : \ 0 < |x_0| \le \delta\}$, $\mathcal{S}_\delta(f)$ be the set of nonextendable solutions of system (1) with initial values $x(0) \in B_\delta$, and $\mathcal{S}_{\delta,x_0}(f) \subset \mathcal{S}_\delta(f)$ – its subset consisting of solutions satisfying the additional condition $x(0) = cx_0$, $c > 0$.

The initial concepts of this report are such properties of the zero solution as stability, asymptotic stability and complete instability. They are massive [6] in the sense that in their description certain conditions are imposed on all solutions starting in a neighborhood of zero. In addition, each of them can be of one of the following three types: Lyapunov, Perron and upper-limit (the last two ones, introduced relatively recently [4, 5], admit contrasting combinations with the first one [3]).

Definition 1 ([7])**.** We say that system (1) (more precisely, its zero solution) has the *Lyapunov, Perron* or, respectively, *upper-limit*:

1) *stability* if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that any solution $x \in \mathcal{S}_\delta(f)$ satisfies the following corresponding requirement

$$\sup_{t\in\mathbb{R}_+} |x(t)| \le \varepsilon, \quad \varliminf_{t\to+\infty} |x(t)| \le \varepsilon, \quad \varlimsup_{t\to+\infty} |x(t)| \le \varepsilon \tag{2}$$

(assuming by default that this solution is defined on the entire ray $\mathbb{R}_+$; otherwise, this requirement is assumed to fail to hold);

2) *asymptotic stability* if: in the *Perron* or *upper-limit* cases – there exists a $\delta > 0$ such that any solution $x \in \mathcal{S}_\delta(f)$ satisfies corresponding requirement (2) for $\varepsilon = 0$, and in the *Lyapunov* case – it has both upper-limit asymptotic stability and Lyapunov stability;

3) *complete instability* if for some $\varepsilon, \delta > 0$ no solution $x \in \mathcal{S}_\delta(f)$ satisfies corresponding requirement (2).

Each of the properties introduced in Definition 1 can be associated with its radial analogue, in which the initial values of the perturbed solutions are taken not from the complete neighborhood of zero, but only along a given ray starting from zero.

Definition 2 ([10])**.** We say that system (1) has the *Lyapunov, Perron* or *upper-limit*:

4) *radial* property: *stability, asymptotic stability* or *complete instability* in the *direction* of nonzero vector $x_0 \in \mathbb{R}^n$ – if it has the corresponding property from points 1–3 of Definition 1 with the replacement of the set $\mathcal{S}_\delta(f)$ in it by the set $\mathcal{S}_{\delta,x_0}(f)$;

5) *total radial* property: *stability, asymptotic stability* or *complete instability* – if it has this property in each direction.

There is a simple, albeit one-sided, logical connection between the properties introduced in points 1–3 of Definition 1 and their total radial analogues from point 5 of Definition 2.

Theorem 1. *If system* (1) *has stability, asymptotic stability or complete instability of the Lyapunov, Perron or upper-limit type, then it has total radial stability, asymptotic stability or, respectively, complete instability of the same type.*

In the one-dimensional case, the statement of Theorem 1 is reversible.

Theorem 2. *If for $n = 1$ system* (1) *has a total radial property of the Lyapunov, Perron or upper-limit type: stability, asymptotic stability or complete instability, – then it has stability, asymptotic stability or, respectively, complete instability of the same type.*

In the special case of a linear homogeneous system

$$\dot{x} = A(t)x \equiv f(t,x), \quad t \in \mathbb{R}_+, \quad x \in G \equiv \mathbb{R}^n, \quad A \in C(\mathbb{R}_+, \operatorname{End}\mathbb{R}^n), \tag{3}$$

total radial properties have some peculiarities.

Theorem 3. *If linear system* (3) *has a total radial property of the Lyapunov, Perron or upper-limit type: asymptotic stability or complete instability, – then it has asymptotic stability or, respectively, complete instability of the same type.*

Theorem 4. *If linear system* (3) *has a total radial stability of the Lyapunov or upper-limit type, then it has stability of both of these types at once.*

Remark 1. The problem of the possibility to extending the statement of Theorem 4 also to similar Perron-type stability properties (separately from Lyapunov and upper-limit) from points 1, 5 of Definitions 1, 2 remains unresolved for now.

In the case in which system (1) does not have some of the initial properties of Definition 1, the question of whether this property holds at least to some extent, becomes meaningful. To answer this question, the following definition introduces characteristics (partly new), which are naturally called measures of these properties and have a probabilistic connotation.

Definition 3 ([8])**.** The *measures of Lyapunov, Perron* or *upper-limit stability* and *instability* of system (1) for $\varkappa = \lambda, \pi, \sigma$ are respectively defined by the formulas

$$\mu_\varkappa(f) = \lim_{\varepsilon\to+0} \varliminf_{\delta\to+0} \frac{\operatorname{mes} \mathrm{M}_\varkappa(f,\varepsilon,\delta)}{\operatorname{mes} B_\delta}\,, \quad \mu_{\overline{\varkappa}}(f) = \lim_{\varepsilon\to+0} \varlimsup_{\delta\to+0} \frac{\operatorname{mes} \mathrm{M}_{\overline{\varkappa}}(f,\varepsilon,\delta)}{\operatorname{mes} B_\delta}\,, \tag{4}$$

and the *measures of asymptotic stability* of the same types are defined respectively by formulas

$$\mu_{\lambda_0}(f) = \lim_{\varepsilon\to+0} \varliminf_{\delta\to+0} \frac{\operatorname{mes}(\mathrm{M}_\sigma(f,0,\delta)\cap \mathrm{M}_\lambda(f,\varepsilon,\delta))}{\operatorname{mes} B_\delta}\,, \quad \mu_{\varkappa_0}(f) = \varliminf_{\delta\to+0} \frac{\operatorname{mes} \mathrm{M}_\varkappa(f,0,\delta)}{\operatorname{mes} B_\delta}\,, \tag{5}$$

where $\varkappa = \pi, \sigma$, $\mathrm{M}_{\overline{\varkappa}}(f,\varepsilon,\delta) \equiv 1 - \mathrm{M}_\varkappa(f,\varepsilon,\delta)$ and $\mathrm{M}_\varkappa(f,\varepsilon,\delta)$ – the set of initial values $x(0)$ of all solutions $x \in \mathcal{S}_\delta(f)$ satisfying the corresponding requirement (2).

The concepts introduced in Definition 3 are correct.

Theorem 5. *For any system* (1)*, all the sets in formulas* (4) *and* (5) *are measurable, and the limits as $\varepsilon \to +0$ exist and can be replaced in the formulas for stability measures (including the Lyapunov asymptotic) and instability, respectively, by the exact lower and upper bounds at $\varepsilon > 0$.*

There is a logical connection between total radial properties and their measures.

Theorem 6. *If system* (1) *has a total radial property of some type: stability, asymptotic stability or complete instability, – then its measure of stability, asymptotic stability or, respectively, instability of this type is equal to 1.*

For some properties, the statement of Theorem 6 can even be slightly strengthened.

Theorem 7. *If system* (1) *has a total radial property of some type: stability or asymptotic stability, – then in the corresponding formula for stability measures* (4) *or, respectively, asymptotic stability measures* (5) *of this type, the lower limit as* $\delta \to +0$ *coincides with the upper limit and is equal to 1.*

Definition 4. We say that system (1) has *Perron* or *upper-limit partial extreme instability* if for any $\delta > 0$ there exists a solution $x \in \mathcal{S}_\delta(f)$ satisfying the corresponding requirement

$$\underline{\lim}_{t\to+\infty} |x(t)| = +\infty, \quad \overline{\lim}_{t\to+\infty} |x(t)| = +\infty \tag{6}$$

(which is assumed to hold, in particular, if the solution is not defined on the entire ray $\mathbb{R}_+$).

Remark 2. The property given in Definition 4 is a type of extreme instability [1, 9]:

– reinforced by the fact that the limit (6) in it is infinite;

– weakened by the fact that requirement (6) is not necessarily satisfied here for all solutions $x \in S_\delta(f)$ but at least for one of them.

Definition 5 ([6])**.** If we assume in Definition 1 that not all solutions $x \in \mathcal{S}_\delta(f)$ satisfy requirement (2) in points 1–3, but *almost* all (i.e. with initial values $x(0)$ from the ball $B_\delta(f)$ minus a subset of measure zero), then the result will be the definition of the following properties of a system (1): *almost stability*, *almost asymptotic stability* and *almost complete instability* of the corresponding type.

The satisfiability of Theorem 6 assumptions for a system (1), does not ensure for it not only stability, asymptotic stability or, respectively, complete instability, but even almost stability, almost asymptotic stability or almost complete instability (in particular [10], for two-dimensional systems).

Theorem 8. *For any natural* $n > 1$ *there exists a system* (1) *with zero linear approximation (along the zero solution) which simultaneously:*

– *has total radial asymptotic stability of all three types;*

– *has both Perron and upper-limit partial extreme instability;*

– *does not have almost stability of any type.*

Theorem 9. *For any natural* $n > 1$ *there exists a system* (1) *with zero linear approximation (along the zero solution) which simultaneously:*

– *has total radial complete instability of all three types;*

– *has both Perron and upper-limit partial extreme instability;*

– *does not have almost complete instability of any type.*

Theorem 10. *For any natural $n > 1$ there exists a system* (1) *with zero linear approximation (along the zero solution) which simultaneously has:*

– *total radial asymptotic stability of all three types;*

– *both Perron and upper-limit partial extreme instability;*

– *both Perron and upper-limit almost asymptotic stability.*

Remark 3. It does not seem possible to strengthen Theorem 10 by adding Lyapunov asymptotic almost stability to its formulation, since the presence of this stability implies the presence of Lyapunov stability in a system (see [3, Theorem 3]), which can not be implemented in a simultaneous combination with Perron partial extreme instability.

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References

[1] A. A. Bondarev, Two contrasting examples of multidimensional differential systems with Lyapunov extreme instability. (Russian) *Mat. Zametki* **115** (2024), no. 1, 24–42; translation in *Math. Notes* **115** (2024), no. 1-2, 21–36.

[2] A. A. Bondarev, Three counterexamples of two-dimensional autonomous differential systems with total radial properties. (Russian) *Differ. Uravn.* **60** (2024), no. 11 (in press).

[3] A. A. Bondarev and I. N. Sergeev, Examples of differential systems with contrasting combinations of Lyapunov, Perron, and upper-limit properties. (Russian) *Dokl. Akad. Nauk* **506** (2022), 25–29; translation in *Dokl. Math.* **106** (2022), no. 2, 322–325.

[4] I. N. Sergeev, Definition and some properties of Perron stability. (Russian) *Differ. Uravn.* **55** (2019), no. 5, 636–646; translation in *Differ. Equ.* **55** (2019), no. 5, 620–630.

[5] I. N. Sergeev, Definition of upper-limit stability and its relationship with Lyapunov stability and Perron stability. (Russian) *Differ. Uravn.* **56** (2020), no. 11, 1556–1557.

[6] I. N. Sergeev, Massive and almost massive properties of stability and instability of differential systems. (Russian) *Differ. Uravn.* **57** (2021), no. 11, 1576–1578.

[7] I. N. Sergeev, On Perron, Lyapunov and upper-limit properties of stability of differential systems. (Russian) *Tr. Semin. Im. I. G. Petrovskogo.* **33** (2023), 353–423.

[8] I. N. Sergeev, Definition and properties of measures of stability and instability of zero solution of a differential system. (Russian) *Mat. Zametki* **113** (2023), no. 6, 895–904; translation in *Math. Notes* **113** (2023), no. 5-6, 831–839.

[9] I. N. Sergeev, Definition and properties of extreme instability of the zero solution of a differential system. (Russian) *Differ. Uravn.* **59** (2023), no. 6, 858–859.

[10] I. N. Sergeev, The relationship between stability and radial stability of a differential system. (Russian) *Differ. Uravn.* **60** (2024), no. 11 (in press).