Asymptotic Representations of Solutions to Differential Equations of the Fourth Order with Nonlinearities, Close to Regularly Varying

M. A. Bilozerova, N. V. Sharai

Odessa I. I. Mechnikov National University, Odessa, Ukraine
E-mails: Marbel@ukr.net; sharay@onu.edu.ua

The differential equation

$$y^{(4)} = \alpha_0 p(t) \prod_{i=0}^{3} \varphi_i(y^{(i)}) \exp\Big(\gamma \Big| \sum_{i=0}^{3} \ln |y^{(i)}| \Big|^{\mu}\Big), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $\gamma \in R$, $\mu \in]0;1[$, $p : [a, \omega[^{1} \to]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$, $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ $(i = 0, 1, 2, 3)$ are the continuous functions, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is either the interval $[y_i^0, Y_i[$ [2] or the interval $]Y_i, y_i^0]$, is considered.

We suppose also that every $\varphi_i(z)$ is regularly varying as $z \to Y_i$ $(z \in \Delta_{Y_i})$ of index σ_i and $\sum\limits_{i=0}^{3} \sigma_i \ne 1$.

According to properties of regularly varying functions (see, for example, the monograph [7]) it is clear that for every defined on $[t_0, \omega[\subset [a, \omega[$ solution y of the equation (1) such that

$$y^{(i)} : [t_0, \omega[\to \Delta_{Y_i}, \quad \lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1, 2, 3), \tag{2}$$

the representations $\varphi_i(y^{(i)}(t)) = |y^{(i)}(t)|^{\sigma_i + o(1)}$ take place as $t \uparrow \omega$. Therefore the equation (1) is in some sense similar to the well known differential equation of Emden–Fowler type.

The first results on the asymptotics of solutions of differential equations with regularly varying nonlinearities have been obtained in the works by V. Marić, M. Tomić [6], S. D. Taliaferro [8], V. M. Evtukhov, L. O. Kirillova [4] and some other authors for the differential equations of the second order of the type

$$y'' = \alpha_0 p(t) \varphi(y).$$

Any regularly varying function is a product of some power function and some slowly varying function. Therefore researches of equations with regularly varying nonlinearities have been connected with the wish to extend to such equations the results, that have been received during the 20th century for the equations with power nonlinearities, in particular, for the generalized equation of Emden–Fowler's type, particular cases of which appear in a lot of sciences of nature.

We call the solution y of the equation (1), that satisfies (2), the $P_\omega(Y_0, Y_1, Y_2, Y_3 \lambda_0)$-solution $(-\infty \le \lambda_0 \le +\infty)$ if the next condition takes place

$$\lim_{t \uparrow \omega} \frac{(y'''(t))^2}{y^{(4)}(t)\, y''(t)} = \lambda_0.$$

[1] If $\omega > 0$, we will take $a > 0$.
[2] If $Y_i = +\infty (Y_i = -\infty)$, we take $y_i^0 > 0$ $(y_i^0 < 0)$, correspondingly.

The improvement of mathematical models of physical phenomena contributed to the growth of the number of results for equations of greater than the general form. In the works by V. M. Evtukhov and A. V. Drozhzhyna (see, for example, [3]) the differential equation of general form

$$y^{(n)} = f(t, y, y', \dots, y^{(n-1)})$$

was investigated. Here $f : [a, \omega[\times \Delta_{Y_0} \times \dots \times \Delta_{Y_{n-1}} \to \mathbb{R}$ is a continuous function, $-\infty < a < \omega \leq +\infty$, $\Delta_{Y_{i-1}}$ is some one-sided neighbourhood of Y_{i-1}, Y_{i-1} equals to zero or to $\pm\infty$, $i = 1, \dots, n$. The subject of the research is $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$-solutions of this equation, conditions of their existence and also asymptotic as $t \uparrow \omega$ representations of such solutions and their derivatives up to the order $n-1$. The class of $P_\omega(Y_0, \dots, Y_{n-1}, \lambda_0)$-solutions was introduced in the works by V. M. Evtukhov and it appeared to be an enough wide class of monotone solutions. It includes regularly, slowly and rapidly varying as $t \uparrow \omega$ solutions and also some types of singular solutions. Every of the mentioned above $n+2$ types of $P_\omega(Y_0, Y_1, \dots, Y_{n-1}, \lambda_0)$-solutions of the differential equation of the n-th order of general form is studied separately by the fulfillment of the condition $(RN)_{\lambda_0}$. The kernel of the condition is the fact that onto any of such solutions the equation is in some sense asymptotically near to the equation

$$y^{(n)} = \alpha_0 p(t) \prod_{j=1}^{n} \varphi_{j-1}(y^{(j-1)}), \tag{3}$$

where $\alpha_0 \in \{-1; 1\}$, $p : [a, \omega[\to]0, +\infty[$ is a continuous function, $\varphi_{j-1} : \Delta_{Y_{j-1}} \to]0, +\infty[$ is a continuous regularly varying function of the order σ_{j-1} as $y^{(j)} \to Y_{j-1}$, $j = 1, \dots, n$.

In the equation (1) the nonlinearity is not near to the form (3) because of the type of the function

$$\exp\Big(\gamma \Big|\sum_{i=0}^{3} \ln|y^{(i)}|\Big|^{\mu}\Big).$$

It follows from the definition of $P_\omega(\lambda_0)$-solution that in cases $\lambda_0 \in \mathbb{R} \setminus \{0, \frac{1}{2}, \frac{2}{3}\}$ every $P_\omega(\lambda_0)$-solution of the equation (1) is regularly varying as $t \uparrow \omega$. In case of second order differential equation for all $P_\omega(\lambda_0)$-solutions of the equation of the type (1) the necessary and sufficient conditions of existence and asymptotic representations as $t \uparrow \omega$ were found (see, for example, [1, 2, 4–6, 8, 8]).

Let us introduce the subsidiary notations.

$$\gamma_0 = 1 - \sum_{j=0}^{n-1} \sigma_j, \quad \mu_n = \sum_{j=0}^{n-1} (n-j-1)\sigma_j, \quad \pi_\omega(t) = \begin{cases} t & \text{if } \omega = +\infty, \\ t - \omega & \text{if } \omega < +\infty, \end{cases}$$

$$\theta_i(z) = \varphi_i(z)|z|^{-\sigma_i}, \quad a_{0i} = (n-i)\lambda_{n-1}^0 - (n-i-1) \ \ (i = 1, \dots, n),$$

$$C = \alpha_0 |\lambda_{n-1}^0 - 1|^{\mu_n} \prod_{k=0}^{n-2} \Big| \prod_{j=k+1}^{n-1} a_{0j} \Big|^{-\sigma_k} \operatorname{sign} y_{n-1}^0,$$

$$I_0(t) = \int_{A_\omega^0}^{t} Cp(\tau)|\pi_\omega(\tau)|^{\mu_n} \, d\tau, \quad I_1(t) = \int_{A_\omega^1}^{t} \alpha_0 p(\tau) \, d\tau,$$

$$A_\omega^0 = \begin{cases} a & \text{if } \displaystyle\int_a^\omega p(\tau)|\pi_\omega(\tau)|^{\gamma_0} \, d\tau = +\infty, \\ \omega & \text{if } \displaystyle\int_a^\omega p(\tau)|\pi_\omega(\tau)|^{\gamma_0} \, dtau < +\infty, \end{cases} \qquad A_\omega^1 = \begin{cases} a & \text{if } \displaystyle\int_a^\omega p(\tau) \, d\tau = +\infty, \\ \omega & \text{if } \displaystyle\int_a^\omega p(\tau) \, d\tau < +\infty, \end{cases}$$

$$J(t)=\int\limits_{B_\omega}^{t}|\gamma_0 I_1(\tau)|^{\frac{1}{\gamma_0}}\,d\tau,\quad B_\omega=\begin{cases} a & \text{if } \int\limits_a^\omega |I_1(\tau)|^{\frac{1}{\gamma_0}}\,d\tau=+\infty,\\ \omega & \text{if } \int\limits_a^\omega |I_1(\tau)|^{\frac{1}{\gamma_0}}\,d\tau<+\infty.\end{cases}$$

The following conclusions take place for the equation (1).

Theorem 1. *The next conditions are necessary for the existence of $P_\omega(Y_0,Y_1,Y_2,Y_3\lambda_0)$-solutions ($\lambda_0\in\mathbb{R}\setminus\{0,1,\frac{1}{2},\frac{2}{3}\}$) of the equation (1):*

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)I_0'(t)}{I_0(t)}=\frac{\gamma_0}{\lambda_{n-1}^0-1},\quad \lim_{t\uparrow\omega} y_i^0|\pi_\omega(t)|^{\frac{a_{0i+1}}{\lambda_{n-1}^0-1}}=Y_i, \tag{4}$$

$$y_i^0 y_{i+1}^0 a_{0i+1}(\lambda_{n-1}^0-1)\pi_\omega(t)>0 \ \text{ as } \ t\in[a,\omega[\,, \tag{5}$$

where $y_3^0=\alpha_0$, $i=0,\dots,3$.

If the equation

$$\sum_{k=0}^{3}\sigma_k\prod_{i=k+1}^{3}a_{0i}\prod_{i=1}^{k}(a_{0i}+\lambda)=(1+\lambda)\prod_{i=1}^{3}(a_{0i}+\lambda)$$

has no roots with zero real part, then the conditions (4), (5) are sufficient for the existence of $P_\omega(Y_0,Y_1,Y_2,Y_3\lambda_0)$-solutions of the equation (1). For any such solution the next asymptotic representations as $t\uparrow\omega$

$$\frac{|y^{(n-1)}(t)|^{\gamma_0}\exp\big(-\gamma|\sum_{i=0}^{3}\ln|y^{(i)}||^{\mu}\big)}{\prod\limits_{j=0}^{n-1}\theta_j(y^{(j)}(t))}=\gamma_0 I_0(t)[1+o(1)],$$

$$\frac{y^{(i)}(t)}{y^{(n-1)}(t)}=\frac{[\,(\lambda_{n-1}^0-1)\pi_\omega(t)]^{n-i-1}}{\prod\limits_{j=i+1}^{n-1}a_{0j}}[1+o(1)],$$

where $i=0,\dots,2$, take place.

By additional conditions on the functions $\varphi_0,\varphi_1,\dots,\varphi_3$ the asymptotic representations as $t\uparrow\omega$ of $P_\omega(Y_0,Y_1,Y_2,Y_3\lambda_0)$-solutions and their derivatives from the first to third order are found in another form.

In order to formulate our following results, we present the next definition.

We call the slowly varying as $z\to Y$ $(z\in\Delta)$ function θ satisfies the condition S if for every continuously differentiable function $L:\Delta\to]0;+\infty[$ such that

$$\lim_{\substack{z\to Y\\ z\in\Delta}}\frac{zL'(z)}{L(z)}=0,$$

the next representation takes place

$$\theta(zL(z))=\theta(z)[1+o(1)] \ \text{ as } \ z\to Y \ \ (z\in\Delta).$$

The next result follows from Theorem 1.

Theorem 2. *Let the functions $\theta_0,\dots,\theta_3$ satisfy the condition S. Then for any $P_\omega(Y_0,Y_1,Y_2,Y_3\lambda_0)$-solution ($\lambda_0\in\mathbb{R}\setminus\{0,1,\frac{1}{2},\frac{2}{3}\}$) of the equation* (1) *the next asymptotic representations as $t\uparrow\omega$*

$$y^{(n-1)}(t)\exp\Big(-\frac{\gamma}{\gamma_0}\Big|\sum_{i=0}^{3}\ln|y^{(i)}|\Big|^{\mu}\Big)=\Big|\gamma_0 I_0(t)\prod_{j=0}^{n-1}\theta_j\big(y_j^0|\pi_\omega(t)|^{\frac{a_{0j+1}}{\lambda_{n-1}^0-1}}\big)\Big|^{\frac{1}{\gamma_0}}\operatorname{sign}y_{n-1}^0[1+o(1)],$$

$$y^{(i)}(t)=y^{(n-1)}(t)\,\frac{[(\lambda_{n-1}^0-1)\pi_\omega(t)]^{n-i-1}}{\prod\limits_{j=i+1}^{n-1}a_{0j}}\,[1+o(1)],\quad i=0,\dots,n-2$$

take place.

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