Regularly Varying Solutions of Differential Equations of the Second Order with Nonlinearities of Exponential Types

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We consider the following differential equation

$$y'' = \alpha_0 p(t) \exp\big(R_0(y, y') + \exp(R_1(y, y'))\big), \tag{1}$$

where $\alpha_0 \in \{-1; 1\}$, $p : [a, \omega[\to]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$, the functions $R_k : \Delta_{Y_0} \times \Delta_{Y_1} \to]0, +\infty[$ $(k \in \{0, 1\})$ are continuously differentiable, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is either $[y_i^0, Y_i[$ [1] or $]Y_i, y_i^0]$. We also suppose the functions R_k satisfy the conditions

$$\lim_{\substack{(y_0,y_1)\to(Y_0,Y_1)\\ (y_0,y_1)\in\Delta_{Y_0}\times\Delta_{Y_1}}} R_k(y_0, y_1) = +\infty, \tag{2}$$

$$\lim_{\substack{y_i\to Y_i\\ y_i\in\Delta_{Y_i}}} \frac{y_i \, \frac{\partial R_k(y_0,y_1)}{\partial y_i}}{R_k(y_0, y_1)} = \gamma_{ki} \ \text{ uniformly by } \ y_j \ne y_i \ \ (k, i, j \in \{0, 1\}). \tag{3}$$

Here functions R_k $(k \in \{0, 1\})$ are in some sense near to regularly varying functions, that are useful for investigations of equations of such a type. Theory of such a functions and their properties are described in the book [4]. Functions that satisfy conditions (2), (3) can be written, for example, as $|y_0|^{\gamma_{k0}}|y_1|^{\gamma_{k1}} \exp(\ln^{\mu}|y_0y_1|)$, $|y_0|^{\gamma_{k0}}|y_1|^{\gamma_{k1}} \ln^{\mu_1}|y_0y_1| \ln\ln|y_0y_1|$, $0 < \mu < 1$, $\mu_1 \in \mathbb{R}$. Differential equations of the second order, containing both power and exponential nonlinearities in the right-hand side, play an important role in the development of the qualitative theory of differential equations. Such equations also have many applications in practice. This happens, for example, when studying the distribution of the electrostatic potential in the cylindrical volume of the plasma of combustion products. The corresponding equation can be reduced to the following

$$y'' = \alpha_0 p(t) e^{\sigma y} |y'|^{\lambda}.$$

In the works by Evtukhov V. M. and Drik N. G. (see, for example, [3]) under certain conditions for the p function, results were obtained about the asymptotic behavior of all correct solutions of this equation. Partial case of the equation (1) was studied in [2].

The solution y to the equation (1) is called $P_\omega(Y_0, Y_1, \lambda_0)$-solution, if

$$y^{(i)} : [t_0, \omega[\to \Delta_{Y_i}, \ \ [t_0, \omega[\subset [a, \omega[, \quad \lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1), \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y''(t)y(t)} = \lambda_0.$$

[1] As $Y_i = +\infty$ $(Y_i = -\infty)$ assume $y_i^0 > 0$ $(y_i^0 < 0)$.

The aim of the work is to establish the necessary and sufficient conditions for the existence to the equation (1) $P_\omega(Y_0, Y_1, \lambda_0)$-solutions and asymptotic representation as $t \uparrow \omega$ for such solutions and its first order derivatives in cases $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$.

To present the results, we introduce the next subsidiary notations.

$$\pi_\omega(t) = \begin{cases} t & \text{as } \omega = +\infty, \\ t - \omega & \text{as } \omega < +\infty, \end{cases}$$

and for every monotone continuously differentiable function $y : [t_0, \omega[\to \Delta_{Y_0}$ such that

$$\lim_{t\uparrow\omega} y(t) = Y_0, \quad \lim_{t\uparrow\omega} y'(t) = Y_1,$$

$$\Phi_0(y(t)) = \int_{Y_0}^{y(t)} \exp\Big(- R_0\big(\tau, y'(t(\tau))\big) - \exp\big(R_1(\tau, y'(t(\tau)))\big)\Big)\, d\tau,$$

where $t(y)$ is the inverse function for $y(t)$,

$$\Phi_1(y) = \int_{Y_0}^{y} \frac{\Phi_0(\tau)}{\tau}\, d\tau, \quad Z_1 = \lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \Phi_1(y),$$

$$I(t) = \alpha_0(\lambda_0 - 1) \int_{B_\omega^0}^{t} \pi_\omega(\tau) p(\tau)\, d\tau, \quad B_\omega^0 = \begin{cases} a & \text{as } \displaystyle\int_a^\omega \pi_\omega(\tau) p(\tau)\, d\tau = +\infty, \\ \omega & \text{as } \displaystyle\int_a^\omega \pi_\omega(\tau) p(\tau)\, d\tau < +\infty, \end{cases}$$

$$I_1(t) = \int_{B_\omega^1}^{t} \frac{\lambda_0 I(\tau)}{(\lambda_0 - 1)\pi_\omega(\tau)}\, d\tau, \quad B_\omega^1 = \begin{cases} a & \text{as } \displaystyle\int_a^\omega \frac{\lambda_0 I(\tau)}{(\lambda_0 - 1)\pi_\omega(\tau)}\, d\tau = +\infty, \\ \omega & \text{as } \displaystyle\int_a^\omega \frac{\lambda_0 |I(\tau)|}{(\lambda_0 - 1)\pi_\omega(\tau)}\, d\tau < +\infty. \end{cases}$$

Remark. It follows from the conditions (2), (3), that functions Φ_0 and Φ_1 are rapidly varying as $y \to Y_0$ ($Y_0 \in \Delta_{Y_0}$) and

$$\lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \frac{\Phi_0''(y) \cdot \Phi_0(y)}{(\Phi_0'(y))^2} = 1, \quad \lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \frac{\Phi_1''(y) \cdot \Phi_1(y)}{(\Phi_1'(y))^2} = 1.$$

The following theorem is obtained.

Theorem 1. *Let $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$, $\gamma_{10}, \gamma_{11} \neq 0$. Then the conditions*

$$\pi_\omega(t) y_1^0 y_0^0 \lambda_0(\lambda_0 - 1) > 0, \quad \pi_\omega(t) y_1^0 \alpha_0(\lambda_0 - 1) > 0 \ \text{ as } \ t \in [a; \omega[\,,$$

$$y_1^0 \cdot \lim_{t\uparrow\omega} |\pi_\omega(t)|^{\frac{1}{\lambda_0 - 1}} = Y_1, \quad \lim_{t\uparrow\omega} I_1(t) = Z_1,$$

$$\lim_{t\uparrow\omega} \frac{I(t)}{\Phi_0(\Phi_1^{-1}(I_1(t)))} = 1, \quad \lim_{t\uparrow\omega} \frac{I_1'(t)\pi_\omega(t)}{\Phi_1'(\Phi_1^{-1}(I_1(t)))\Phi_1^{-1}(I_1(t))} = \frac{\lambda_0}{\lambda_0 - 1}\,,$$

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t) I_1'(t)}{I_1(t)} = \infty, \quad \lim_{t\uparrow\omega} \frac{I'(t)\pi_\omega(t)\Phi_0(\Phi_1^{-1}(I_1(t)))}{\Phi_0'(\Phi_1^{-1}(I_1(t)))\Phi_1^{-1}(I_1(t)) I(t)} = \frac{\lambda_0}{\lambda_0 - 1}\,,$$

are necessary and sufficient for the existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions to the equation (1). *Moreover, the equation* (1) *has a one-parametric family of solutions in case* $(\lambda_0 - 1)\beta < 0$ *and two-parametric family of solutions in cases*

$$\big((1 < \lambda_0 < 3) \wedge (\beta > 0)\big) \vee (\lambda_0 > 3) \wedge (\beta < 0).$$

For every such solution the next asymptotic representations take place as $t \uparrow \omega$

$$y(t) = \Phi_1^{-1}(I_1(t))[1 + o(1)], \quad y'(t) = \frac{\lambda_0}{(\lambda_0 - 1)} \cdot \frac{\Phi_1^{-1}(I_1(t))}{\pi_\omega(t)}\,[1 + o(1)].$$

The differential equation

$$y'' = \alpha_0 p(t) \exp(R_0(y, y')), \tag{4}$$

where $\alpha_0 \in \{-1; 1\}$, $p : [a, \omega[\to]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$, the function $R : \Delta_{Y_0} \times \Delta_{Y_1} \to]0, +\infty[$ is continuously differentiable, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is either $[y_i^0, Y_i[$ [1] or $]Y_i, y_i^0]$, was considered in [1] and it is the equation of type (1), where $\gamma_{10} = \gamma_{11} = 0$.

In this case we have

$$\Phi_0(y) = \int_{Y_0}^{y} \exp\big(-R(\tau, y'(t^{-1}(\tau)))\big)\, d\tau,$$

where $t^{-1}(y)$ is the inverse function for $y(t)$,

$$\Phi_1(y) = \int_{Y_0}^{y} \frac{\Phi_0(\tau)}{\tau}\, d\tau, \quad Z_1 = \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \Phi_1(y).$$

For the equation (4) the next result is valid.

Theorem 2. *Let* $\gamma_0\lambda_0 + \gamma_1 \in \mathbb{R} \setminus \{0, \lambda_0\}$. *Then the conditions*

$$\pi_\omega(t) y_1^0 y_0^0 \lambda_0(\lambda_0 - 1) > 0, \quad \pi_\omega(t) y_1^0 \alpha_0(\lambda_0 - 1) > 0, \ \ t \in [a; \omega[\,,$$

$$y_1^0 \cdot \lim_{t \uparrow \omega} |\pi_\omega(t)|^{\frac{1}{\lambda_0 - 1}} = Y_1, \quad \lim_{t \uparrow \omega} I_1(t) = Z_1,$$

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t)\Big(\frac{I_1(t)}{I_1'(t)}\Big)'}{\frac{I_1(t)}{I_1'(t)}} = \frac{\lambda_0\gamma_0 + \gamma_1 + 1}{\lambda_0 - 1}\,, \quad \lim_{t \uparrow \omega} \frac{I_1''(t) I_1(t)}{(I_1'(t))^2} = 1,$$

$$\lim_{t \uparrow \omega} \frac{I_1'(t)\pi_\omega(t)}{\Phi_1'(\Phi_1^{-1}(I_1(t)))\Phi_1^{-1}(I_1(t))} = \frac{\lambda_0}{\lambda_0 - 1}\,, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t) I_1''(t)}{I_1'(t)} = \infty$$

are necessary and sufficient for the existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions to the equation (4) *in cases* $\lambda_0 \in \mathbb{R} \setminus \{0, 1\}$. *Moreover, for every such solution the next asymptotic representations take place as* $t \uparrow \omega$

$$\Phi_1(y(t)) = I_1(t)[1 + o(1)], \quad \frac{y'(t)\Phi_1'(y(t))}{\Phi_1(y(t))} = \frac{I_1'(t)}{I_1(t)}\,[1 + o(1)].$$

For equations of more concrete type we can find more precise representations. For $t \in [2, +\infty[$ let us consider the differential equation

$$y'' = \frac{1}{4}\, t^{-3} L(t) e^{|y|^4 - t^8} |y'|^3, \tag{5}$$

where $L : [2, +\infty[\to]0, +\infty[$ is slowly varying on infinity function.

In this case

$$I_1(t)=\frac{\lambda_0}{8\sqrt{|\lambda_0-1|}(\lambda_0-1)}\,t^{-14}(L(t))^{-\frac{1}{2}}e^{\frac{1}{2}t^8}[1+o(1)],$$
$$\Phi_0(y)=\frac{1}{2y^3}\,e^{\frac{1}{2}|y|^4}\operatorname{sign}y[1+o(1)] \ \text{ as } \ y\to+\infty,$$
$$\Phi_1(y)=\frac{1}{4y^7}\,e^{\frac{1}{2}|y|^4}[1+o(1)] \ \text{ as } \ y\to+\infty,$$
$$\frac{\Phi_1'(y)}{\Phi_1(y)}=2y^3[1+o(1)] \ \text{ as } \ y\to+\infty.$$

We have that $P_{+\infty}(Y_0,Y_1,\lambda_0)$-solutions of the equation (5) can be only $P_{+\infty}(+\infty,+\infty,2)$-solutions. Moreover, for every such solution the next asymptotic representations take place as $t\to\infty$,

$$\frac{1}{y^7(t)}e^{\frac{1}{2}y^4(t)}=t^{-14}(L(t))^{-\frac{1}{2}}e^{\frac{1}{2}t^8}[1+o(1)],$$
$$y'(t)y^3(t)=2t^7[1+o(1)] \ \text{ as } \ t\uparrow\omega.$$

References

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