Autonomous Seminonlinear Boundary Value Problems with Switchings at Non-Fixed Times

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We study the problem of constructing solutions [3, 4, 7]

$$z(\,\cdot\,,\varepsilon) \in \mathbb{C}^1\big\{[0,T] \setminus \{\tau(\varepsilon)\}_I\big\} \cap \mathbb{C}[0,T], \ \ z(t,\,\cdot\,) \in \mathbb{C}[0,\varepsilon_0]$$

of the autonomous boundary value problem for the equation

$$z'(t,\varepsilon) = A\,z(t,\varepsilon) + \varepsilon\,Z(z(t,\varepsilon),\varepsilon), \quad \ell z(\,\cdot\,,\varepsilon) = 0, \tag{1}$$

which are continuous at $t = \tau(\varepsilon)$. At the point $t = \tau(\varepsilon)$: $0 < \tau(\varepsilon) < T$, its $\tau(0) := \tau_0$ the solution of the boundary value problem (1) might have a limited discontinuity of first derivative [3, 7]. The solution of the boundary value problem (1) is found in a small neighbourhood of the solution

$$z_0(t) \in \mathbb{C}\big\{[0,T] \setminus \{\tau_0\}_I\big\} \cap \mathbb{C}[0,T]$$

of the generating boundary value problem

$$z_0'(t) = A\,z_0(t), \quad \ell z_0(\,\cdot\,) = 0. \tag{2}$$

At the point $t = \tau_0$, the solution of the boundary value problem (2) might have a limited discontinuity of the derivative. Here, $A \in \mathbb{R}^{n\times n}$ is a constant matrix, $Z(z,\varepsilon)$ is a nonlinear vector function, piecewise analytic in the unknown z in a small neighbourhood of the solution of the generating problem (2) and piecewise analytic in a small parameter ε on the interval $[0,\varepsilon_0]$. In addition,

$$\ell z(\,\cdot\,,\varepsilon) := \begin{pmatrix} z(0,\varepsilon) - z(T,\varepsilon) \\ z(\tau(\varepsilon)+0,\varepsilon) - z(\tau(\varepsilon)-0,\varepsilon) \end{pmatrix} = 0, \quad \ell z_0(\,\cdot\,) := \begin{pmatrix} z_0(0) - z_0(T) \\ Z_0(\tau_0+0) - z_0(\tau_0-0) \end{pmatrix} = 0$$

are linear bounded vector functionals. The condition for the solvability of the autonomous nonlinear boundary value problem (1) with switchings leads to the equation

$$F_0(c_0,\tau_0) := P_{Q_r^*}\,\ell K\big[Z(z_0(s,c_0),0);\tau_0\big](\,\cdot\,) = 0. \tag{3}$$

The necessary conditions for the existence of a solution to the autonomous nonlinear boundary value problem (1) with switchings in the critical case are given by the following lemma.

Lemma. *Suppose that there is the critical case* (2) *for the generating boundary value problem. In this case, the generating problem* (2) *has a one-parameter family of solutions* $z_0(t,c_0)$. *Suppose that an autonomous nonlinear boundary value problem* (1) *with switchings at non-fixed times in the neighbourhood of the generating solution* $z_0(t,c_0)$ *has the solution*

$$z(\,\cdot\,,\varepsilon)\in\mathbb{C}\big\{[0,T]\setminus\{\tau(\varepsilon)\big\}_I\big\}\cap\mathbb{C}[0,T],\ \ z(t,\,\cdot\,)\in\mathbb{C}[0,\varepsilon_0].$$

Under these conditions, the equality (3) *holds.*

The equation (3), will be further called the equation for the generating constants of the boundary value problem (1) with switchings in the critical case. Let us assume that the equation for the generating constants (3) of the boundary value problem (1) with switchings has real roots. Fixing one of the real solutions

$$c_0^*\in\mathbb{R}^r,\ \ \tau_0^*\in\mathbb{R}$$

of the equation (3) we get the problem of constructing a solution of the nonlinear boundary value problem (1) in a small neighbourhood of the solution $z_0(t,c_0^*)=X_r(t)\,c_0^*$, $c_0^*\in\mathbb{R}^r$, of the generating boundary value problem (2). The traditional condition for the solvability of a boundary value problem (1) with switchings in a small neighbourhood of the solution of the generating problem is the requirement [3]

$$P_{B_0^*}P_{Q_r^*}\neq 0,\quad B_0:=F'_{\check{c}_0}(c_0^*,\tau_0^*)\in\mathbb{R}^{r\times(r+1)},\quad \check{c}_0:=\begin{pmatrix}c_0 & \tau_0\end{pmatrix}^*, \tag{4}$$

where $P_{B_0^*}:\mathbb{R}^r\to\mathbb{N}(B_0^*)$ is an orthoprojector matrix [3]. The solution of the boundary value problem (1) with switchings is given by

$$z(t,\varepsilon):=z_0(t,c_0^*)+u_1(t,\varepsilon)+\cdots+u_k(t,\varepsilon)+\cdots,\ \ \tau(\varepsilon)=\tau_0^*+\xi_1(\varepsilon)+\xi_2(\varepsilon)+\cdots+\xi_k(\varepsilon)+\cdots.$$

The nonlinear vector function $Z(z(t,\varepsilon),\varepsilon)$ is analytical with respect to the unknown $z(t,\varepsilon)$ in a small neighbourhood of the solution of the generating boundary value problem (2) and the constant τ_0^*, therefore in the given neighbourhood there exist an expansion

$$Z(z(t,\varepsilon),\varepsilon)=Z_0(z_0(t,c_0^*),\varepsilon)+Z_1\big(z_0(t,c_0^*),u_1(s,\varepsilon),\varepsilon\big)+Z_2\big(z_0(t,c_0^*),u_1(s,\varepsilon),u_2(s,\varepsilon),\varepsilon\big)+\cdots.$$

The first approximation to the solution of the nonlinear periodic boundary value problem (1) in the critical case

$$z_1(t,\varepsilon)=z_0(t,c_0^*)+u_1(t,\varepsilon),\quad \tau_1(\varepsilon)=\tau_0^*+\xi_1(\varepsilon),$$
$$u_1(t,\varepsilon)=X_r(t)c_1(\varepsilon)+\varepsilon\,G\big[Z_0(z_0(s,c_0^*),z_0'(s,c_0^*),\varepsilon);\tau_0^*\big](t)$$

determines the solution of the nonlinear periodic boundary value problem of the first approximation

$$u_1'(t,\varepsilon)=A\,u_1(t,\varepsilon)+\varepsilon\,Z_0(z_0(t,c_0^*),\varepsilon),\ \ \ell u_1(\,\cdot\,,\varepsilon)=0.$$

The matrix B_0, which is the key matrix in the study of the boundary value problem (1), takes the form

$$B_0=P_{Q_r^*}\,\ell K\big[\mathcal{A}_0(s)X_r(s);1\big](\,\cdot\,);\quad \mathcal{A}_0(t)=\frac{\partial Z(z(t,\varepsilon),\varepsilon)}{\partial z(t,\varepsilon)}\bigg|_{\substack{z(t,\varepsilon)=z_0(t,c_0^*)\\ \varepsilon=0}}.$$

The second approximation to the solution of the nonlinear periodic boundary value problem (1), in the critical case

$$z_2(t,\varepsilon):=z_0(t,c_0^*)+u_1(t,\varepsilon)+u_2(t,\varepsilon),\quad \tau_2(\varepsilon)=\tau_0^*+\xi_1(\varepsilon)+\xi_2(\varepsilon),$$

determines the solution of the nonlinear periodic boundary value problem of the second approximation

$$u_2'(t,\varepsilon) = A\,u_2(t,\varepsilon) + \varepsilon\, Z_1(z_0(t,c_0^*), u_1(t,\varepsilon), \varepsilon), \quad \ell u_2(\,\cdot\,,\varepsilon) = 0.$$

The condition of solvability of the boundary value problem of the second approximation

$$F_1(c_1(\varepsilon), \xi_1(\varepsilon)) := P_{Q_d^*} \ell K\big[Z_1\big(z_0(s,c_0^*), u_1(s,\varepsilon), z_0'(s,c_0^*), u_1(s,\varepsilon),\varepsilon\big); \xi_1(\varepsilon)\big](\,\cdot\,) = 0$$

is the linear equation

$$F_1(c_1(\varepsilon), \xi_1(\varepsilon)) = B_0\, \check{c}_1(\varepsilon) + \gamma_1(\varepsilon) = 0, \quad \check{c}_1(\varepsilon) := \begin{pmatrix} c_1(\varepsilon) & \xi_1(\varepsilon) \end{pmatrix}^*,$$

which has solutions in case (4), where

$$\gamma_1(\varepsilon) := F_1(\check{c}_1(\varepsilon)) - B_0\, \check{c}_1(\varepsilon).$$

Indeed, let us denote the vector-functions

$$v(t,\varepsilon,\mu) := z_0(t,c_0^*) + \mu\, u_1(t,\varepsilon) + \cdots + \mu^k\, u_k(t,\varepsilon) + \cdots\,,$$
$$g(\varepsilon,\mu) := \tau_0^* + \mu\, \xi_1(\varepsilon) + \mu^2\, \xi_2(\varepsilon) + \cdots + \mu^k\, \xi_k(\varepsilon) + \cdots\,,$$

while

$$F_1(c_1(\varepsilon), \xi_1(\varepsilon)) = P_{Q_d^*} \ell K\big[Z_1(z_0(s,c_0^*), u_1(s,\varepsilon), \varepsilon); \xi_1(\varepsilon)\big](\,\cdot\,)$$
$$= P_{Q_d^*}\, \ell K\big[Z_\mu'(v(t,\varepsilon,\mu),\varepsilon); g_\mu'(\varepsilon,\mu)\big](\,\cdot\,)\Big|_{\mu=0} = P_{Q_d^*}\, \ell K\big[\mathcal{A}_0(s) u_1(s,\varepsilon); \xi_1(\varepsilon)\big](\,\cdot\,),$$

therefore

$$B_0 := F_{\check{c}_1(\varepsilon)}'(\check{c}_1(\varepsilon)) \in \mathbb{R}^{r\times(r+1)}.$$

Thus, under the condition (4), we obtain at least one solution to the first approximation boundary value problem

$$z_1(t,\varepsilon) := z_0(t,c_0^*) + u_1(t,\varepsilon), \quad \tau_1(\varepsilon) = \tau_0^* + \xi_1(\varepsilon), \quad \check{c}_1(\varepsilon) = -B_0^+\, \gamma_1(\varepsilon),$$
$$u_1(t,\varepsilon) = X_r(t) c_1(\varepsilon) + \varepsilon\, G\big[Z_1(z_0(s,c_0^*), u_1'(s,\varepsilon), \varepsilon); \xi_1(\varepsilon)\big](t).$$

The conditions for solvability of boundary value problems of the following approximations

$$F_j(\check{c}_j(\varepsilon)) := P_{Q_d^*}\, \ell K\big[Z_j(z_0(s,c_0^*), u_1(t,\varepsilon), \ldots,\ u_j(s,\varepsilon), \xi_j(\varepsilon), \varepsilon)\big](\,\cdot\,) = 0$$

are linear equations

$$F_j(\check{c}_j(\varepsilon)) = B_0\, \check{c}_j(\varepsilon) + \gamma_j(\varepsilon) = 0, \quad j = 1, 2, \ldots, k,$$

where

$$B_0 = F'(\check{c}_j(\varepsilon)), \quad \gamma_j(\varepsilon) := F(\check{c}_j(\varepsilon)) - B_0\, \check{c}_j(\varepsilon), \quad j = 1, 2, \ldots, k.$$

In the case (4), the last equation has solutions. The sequence of approximations to the solution of the nonlinear periodic boundary value problem (1) in the critical case is determined by the iterative scheme

$$\begin{gathered}
z_1(t,\varepsilon) = z_0(t,c_0^*) + u_1(t,\varepsilon), \quad \tau_1(\varepsilon) = \tau_0^* + \xi_1(\varepsilon), \quad \check{c}_1(\varepsilon) = -B_0^+\, \gamma_1(\varepsilon), \\
u_1(t,\varepsilon) = X_r(t) c_1(\varepsilon) + \varepsilon\, G\big[Z_1(z_0(s,c_0^*), u_1'(s,\varepsilon), \varepsilon); \xi_1(\varepsilon)\big](t); \\
z_2(t,\varepsilon) = z_0(t,c_0^*) + u_1(t,\varepsilon) + u_2(t,\varepsilon), \quad \tau_2(\varepsilon) = \tau_0^* + \xi_1(\varepsilon) + \xi_2(\varepsilon), \\
u_2(t,\varepsilon) = X_r(t) c_2(\varepsilon) + \varepsilon\, G\big[Z_2(z_0(s,c_0^*), u_1(s,\varepsilon), u_2(s,\varepsilon), \varepsilon); \xi_2(\varepsilon)\big](t); \\
z_{k+1}(t,\varepsilon) = z_0(t,c_0^*) + u_1(t,\varepsilon) + \cdots + u_{k+1}(t,\varepsilon), \\
\tau_{k+1}(\varepsilon) = \tau_0^* + \xi_1(\varepsilon) + \cdots + \xi_{k+1}(\varepsilon), \\
u_{k+1}(t,\varepsilon) = X_r(t)\, c_{k+1}(\varepsilon) + \varepsilon\, G\big[Z_k(z_0(s,c_0^*), u_1(s,\varepsilon), \ldots, u_k(s,\varepsilon), \varepsilon); \xi_k(\varepsilon)\big](t), \\
k = 0, 1, 2, \ldots\,.
\end{gathered} \tag{5}$$

Theorem. *Suppose that there is the critical case of the generating boundary value problem* (2)*. In this case, the generating problem* (2) *has a family of solutions*

$$z_0(t,c_0)=X_r(t)\,c_0,\ \ c_0\in\mathbb{R}^r.$$

In the case of (4) *in the small neighbourhood of the generating solution* $z_0(t,c_0^*)$ *and the constant* τ_0^* *the problem* (1) *with switchings has at least one solution. The sequence of approximations to the solution*

$$z(\,\cdot\,,\varepsilon)\in\mathbb{C}\big\{[0,T]\setminus\{\tau(\varepsilon)\}_I\big\}\cap\mathbb{C}[0,T],\ \ z(t,\,\cdot\,)\in\mathbb{C}[0,\varepsilon_0]$$

of the autonomous boundary value problem (1) *with switchings is determined by an iterative scheme* (5)*. If there exist constants* $0<\gamma<1$*, and* $0<\delta<1$ *such that inequalities hold*

$$\begin{aligned}&\|u_1(t,\varepsilon)\|\leq\gamma\|z_0(t,c_0^*)\|,\quad \|u_{k+1}(t,\varepsilon)\|\leq\gamma\|u_k(t,\varepsilon)\|,\\ &|\xi_1(\varepsilon)|\leq\delta\,|\tau_0^*|,\quad |\xi_{k+1}(\varepsilon)|\leq\delta\,|xi_k(\varepsilon)|,\ \ k=1,2,\dots,\end{aligned}\tag{6}$$

then the iterative scheme (5) *converges to the solution of the autonomous boundary value problem* (1) *with switchings.*

The obtained iterative scheme is applied to find approximations to the periodic solution of the equation with switchings at non-fixed moments of time, which models a nonisothermal chemical reaction [1, 2].

The obtained convergence condition (6) of the iterative scheme (5) allows us to estimate the interval of values of the small parameter $\varepsilon\in[0,\varepsilon_0]$, $0\leq\varepsilon_*\leq\varepsilon_0$, for which the convergence of the iterative scheme (5) is preserved, different from similar estimates [5, 6].

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