On Oscillation of Solutions to One Neutral Type Differential Equation

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Consider a second-order differential equation of neutral type with constant delays

$$(y - py_\tau)'' + q(t)f(y_\sigma) = 0, \quad y_\rho(t) \equiv y(t-\rho), \;\; t \in [t_0, +\infty), \tag{1}$$

where $0 < p < 1$, $\tau, \sigma > 0$, $q \in C[t_0, +\infty)$, $q \geqslant 0$.

Denote $\rho \equiv \max\{\tau, \sigma\}$.

Definition 1. The solution to equation (1) is the function $y \in C[t_0 - \rho, +\infty)$, satisfying this equation, such that $y - py_\tau \in C^2[t_0, +\infty)$.

Definition 2. The solution y of equation (1) is called oscillatory if for any $t_1 \geqslant t_0$ there exists $t_2 > t_1$ such that $y(t_2) = 0$.

Definition 3. We will say that a function f such that $f'(y) \geqslant 0$, $y \in \mathbb{R}$, and $yf(y) > 0$, $y \neq 0$, satisfies:

- the superlinear condition, if for any $\varepsilon > 0$ the inequalities hold:

$$0 < \int\limits_{\varepsilon}^{+\infty} \frac{dy}{f(y)} < +\infty, \quad 0 < -\int\limits_{-\infty}^{-\varepsilon} \frac{dy}{f(y)} < +\infty;$$

- the sublinear condition, if for any $\varepsilon > 0$ the inequalities hold:

$$0 < \int\limits_{0}^{\varepsilon} \frac{dy}{f(y)} < +\infty, \quad 0 < -\int\limits_{-\varepsilon}^{0} \frac{dy}{f(y)} < +\infty.$$

In the case $p = \tau = \sigma = 0$ and $f(y) = |y|^\gamma \operatorname{sgn} y$, equation (1) is an Emden–Fowler type equation

$$y'' + q(t)|y|^\gamma \operatorname{sgn} y = 0. \tag{2}$$

The following criteria for the oscillation of all its solutions are known.

Theorem A (Atkinson [2])**.** *If $q \in C[0, +\infty)$, $q \geqslant 0$ and $\gamma = 2n - 1$, $n \in \mathbb{N}$, $n > 1$, then all solutions to equation* (2) *are oscillatory iff*

$$\int\limits_{0}^{+\infty} tq(t)\, dt = +\infty.$$

Theorem B (Belohorec [3]). *If $q_j \in C[0,+\infty)$, $q_j \geqslant 0$ and $\gamma_j = p_j/r_j \in (0,1)$, where p_j, r_j – natural, odd and $j \in \mathbb{N}$, then all solutions of the equation $y'' + \sum\limits_{j=1}^{n} q_j(t)y^{\gamma_j} = 0$ are oscillatory iff*

$$\int\limits_0^{+\infty} \sum_{j=1}^{n} t^{\gamma_j} q_j(t)\,dt = +\infty.$$

A strengthening of Atkinson's theorem for all real $\gamma > 1$ was proven in [4], the oscillation of solutions of high-order Emden–Fowler type equations was studied in [5]. A more general case of equation (2) was considered in [1].

In [6] criteria for the oscillation of all solutions of equation (1) in the cases of superlinearity and sublinearity of the function f are proved. The following results complement and clarify these criteria.

Lemma 1. *Let y be the solution of equation (1) such that $y > 0$ for every $t \geq t_0 \geq 0$ and $z = y - py_\tau$. Then for every $t \geq t_1$, where $t_1 \geq t_0 + \rho$ is sufficiently large, one of the conditions holds:*

1) $z'' \leq 0$, $z' > 0$, $z < 0$;

2) $z'' \leq 0$, $z' > 0$, $z > 0$.

Moreover, the first condition is satisfied when $\lim\limits_{t\to+\infty} y(t) = 0$. Otherwise, the second condition is true.

Lemma 2. *For every continuous function φ, defined on the segment $[t_0 - \rho, t_0]$, equation (1) has a solution y, extendable to the interval $[t_0, +\infty)$ and satisfying the initial conditions $y(t) = \varphi(t)$ for $t \in [t_0 - \rho, t_0]$.*

Theorem 1. *Let the function $f \in C^1(\mathbb{R})$ be superlinear. Then:*

1) *if $\int\limits_{t_0}^{+\infty} tq(t)\,dt = +\infty$, then all not vanishing at infinity solutions to equation (1) are oscillatory;*

2) *if all solutions to equation (1) are oscillatory, then $\int\limits_{t_0}^{+\infty} tq(t)\,dt = +\infty$.*

Proof. 1) Let y be a non-vanishing non-oscillatory solution to equation (1). Then, due to $yf(y) > 0$, without loss of generality we can assume that $y > 0$ for all $t \geq t_0 \geq 0$. By Lemma 1 for $z = y - py_\tau \geq y$ we have $z'' \leq 0$, $z' > 0$, $z > 0$ for all $t \geq t_1$.

Then

$$0 = z''(t) + q(t)f(y_\sigma(t)) \geq z''(t) + q(t)f(z_\sigma(t)).$$

Let

$$w(t) = \frac{tz'(t)}{f(z_\sigma(t))} \geq 0.$$

We obtain

$$w'(t) + tq(t) \leq \frac{z'(t)}{f(z_\sigma(t))} - \frac{tf'(z_\sigma(t))z'(t)}{[f(z_\sigma(t))]^2}\, z'_\sigma(t) \leq \frac{z'(t)}{f(z_\sigma(t))}\,.$$

Let's integrate the inequality

$$w(t) - w(t_1) + \int\limits_{t_1}^{t} sq(s)\,ds \le \int\limits_{t_1}^{t} \frac{z'(s)}{f(z_\sigma(s))}\,ds \le \int\limits_{t_1}^{t} \frac{z'_\sigma(s)}{f(z_\sigma(s))}\,ds,$$

$$w(t) - w(t_1) + \int\limits_{t_1}^{t} sq(s)\,ds \le \int\limits_{z_\sigma(t_1)}^{z_\sigma(t)} \frac{dv}{f(v)}\,,$$

$$\int\limits_{t_1}^{t} sq(s)\,ds \le w(t_1) + \int\limits_{z_\sigma(t_1)}^{\infty} \frac{dv}{f(v)} = const < +\infty.$$

Tending t to infinity, we arrive at a contradiction.

2) See [6]. □

Remark. The divergence of the integral $\int\limits_0^{+\infty} tq(t)\,dt$ does not guarantee (contrary to the statement from [6]) the oscillation of all solutions to equation (1). For example, the function $y(t) = e^{-t}$ is a particular solution to the equation

$$\Big(y - \frac{1}{2}\,y_1\Big)'' + \Big(\frac{e}{2} - 1\Big)e^{2t-3}y_1^3 = 0,$$

and $\lim\limits_{t\to+\infty} y(t) = 0$ and $\int\limits_0^{+\infty} tq(t)\,dt = +\infty$, where $q(t) \equiv t(e/2-1)e^{2t-3}$.

Theorem 2. *Let the function $f \in C(\mathbb{R})$ be sublinear and $f(uv) \geqslant f(u)f(v)$ for $uv \geqslant 0$. Then:*

1) *if $\int\limits_{t_0}^{+\infty} f(t)q(t)\,dt = +\infty$, then all not vanishing at infinity solutions to equation (1) are oscillatory;*

2) *if all solutions to equation (1) are oscillatory, then $\int\limits_{t_0}^{+\infty} f(t)q(t)\,dt = +\infty$.*

Proof. 1) Let y be a non-vanishing non-oscillatory solution to equation (1). Then, due to $yf(y) > 0$, without loss of generality we can assume that $y > 0$ for all $t \ge t_0 \ge 0$. By Lemma 1 for $z = y - py_\tau \ge y$ we have $z'' \le 0$, $z' > 0$, $z > 0$ for all $t \ge t_1$.

We have

$$0 = z''(t) + q(t)f(y_\sigma(t)) \ge z''(t) + q(t)f(z_\sigma(t)).$$

Since

$$z(t) = z(t_1) + \int\limits_{t_1}^{t} z'(s)\,ds \ge z'(t)(t - t_1),$$

then

$$f(z_\sigma(t)) \ge f(z'_\sigma(t)(t - \sigma - t_1)).$$

For any $\lambda \in (0;1)$, if $t_2 \ge t_1$ is sufficiently large, $t - \sigma - t_2 \ge \lambda t$ for all $t \ge t_2$. Therefore,

$$f\big(z'_\sigma(t)(t - \sigma - t_1)\big) \ge f(\lambda z'_\sigma(t)t) \ge f(\lambda z'_\sigma(t))f(t)$$

and

$$\frac{z''(t)}{f(\lambda z'_\sigma(t))} + q(t)f(t) \le 0.$$

Integrating the resulting inequality, we obtain

$$\int\limits_{t_2}^{t} \frac{z''(s)}{f(\lambda z'_\sigma(s))}\,ds + \int\limits_{t_2}^{t} q(s)f(s)\,ds \le 0,$$

$$\int\limits_{t_1}^{t} q(s)f(s)\,ds \le -\int\limits_{t_2}^{t} \frac{z''(s)}{f(\lambda z'_\sigma(s))}\,ds \le -\int\limits_{t_2}^{t} \frac{z''(s)}{f(\lambda z'(s))}\,ds,$$

$$\int\limits_{t_2}^{t} q(s)f(s)\,ds \le \int\limits_{\lambda z'(t)}^{\lambda z'(t_2)} \frac{dv}{\lambda f(v)} = \int\limits_{0}^{\lambda z'(t_2)} \frac{dv}{\lambda f(v)} - \int\limits_{0}^{\lambda z'(t)} \frac{dv}{\lambda f(v)}\,.$$

Then, by the property of sublinearity of the function f we have

$$\int\limits_{t_2}^{t} q(s)f(s)\,ds \le const < +\infty.$$

Tending t to infinity, we arrive at a contradiction.

2) See [6]. □

Theorem 3. *If the function $f \in C(\mathbb{R})$ is sublinear, $\sigma > \tau$ and $\int\limits_{t_0}^{+\infty} q(t)\,dt = +\infty$, then all solutions to equation* (1) *are oscillatory.*

Proof. Let y be a non-oscillating solution to (1). Then, due to $yf(y) > 0$, without loss of generality we can assume that $y > 0$ for all $t \ge t_0 \ge 0$.

Let us show that both cases described in Lemma 1 are impossible.

1) If $z > 0$ for all $t \ge t_1$, where $t_1 \ge t_0 + \rho$, we have

$$z = y - py_\tau \ge y.$$

Due to $f' \ge 0$ and equation (1), we obtain

$$z''(t) + q(t)f(z_\sigma(t)) \le 0.$$

Integrating this inequality on the interval $[t_1, t]$, we get

$$\int\limits_{t_1}^{t} q(s)f(z_\sigma(s))\,ds \le z'(t_1),$$

$$\int\limits_{t_1}^{t} q(s)\,ds \le \frac{z'(t_1)}{f(z_\sigma(t_1))} \le \frac{z'(t_1)}{f(z(t_1))} = const < +\infty.$$

Tending t to infinity, we come to a contradiction.

2) If $z<0$ for all $t\ge t_1\ge t_0+\rho$, then

$$z(t)=y(t)-py_\tau(t)<-py_\tau(t),$$
$$y_\sigma(t)<-\frac{z_{\sigma-\tau}(t)}{p}\,.$$

Then, since f is increasing, from equation (1) we have

$$z''(t)+q(t)f\Big(-\frac{z_{\sigma-\tau}(t)}{p}\Big)\le 0.$$

Let us integrate this inequality on the interval $[t-\sigma+\tau,t]$.

$$z'_{\sigma-\tau}(t)-z'(t)+\int\limits_{t-\sigma+\tau}^{t} q(s)f\Big(-\frac{z_{\sigma-\tau}(t)}{p}\Big)p\,ds\le 0.$$

Taking into account the fact that z is positive and increasing, we have

$$-z'_{\sigma-\tau}(t)\Big/f\Big(-\frac{z_{\sigma-\tau}(t)}{p}\Big)+\int\limits_{t-\sigma+\tau}^{t} q(s)\,ds\le 0.$$

Let $w(t)\equiv -z_{\sigma-\tau}(t)/p$. Integrating the inequality on $[t_2,t_3]$, we obtain

$$p\int\limits_{w(t_2)}^{w(t_3)}\frac{dw}{f(w)}+\int\limits_{t_2}^{t_3}\int\limits_{t-\sigma+\tau}^{t} q(s)\;ds\,dt\le 0,$$
$$\int\limits_{t_2}^{t_3}\int\limits_{t-\sigma+\tau}^{t} q(s)\;ds\,dt\le p\int\limits_{0}^{w(t_2)}\frac{dt}{w(t)}-p\int\limits_{0}^{w(t_3)}\frac{dt}{w(t)}\,,$$
$$\int\limits_{t_2}^{t_3}\int\limits_{t-\sigma+\tau}^{t} q(s)\;ds\,dt\le p\int\limits_{0}^{w(t_2)}\frac{dt}{w(t)}\,.$$

Due to the sublinearity of the function f, we get

$$\int\limits_{t_2}^{\infty}\int\limits_{t-\sigma+\tau}^{t} q(s)\;ds\,dt<+\infty,$$

which contradicts the condition $\int\limits_{t_0}^{\infty} q(t)dt=+\infty$. □

References

[1] I. V. Astashova, On existence of nonoscillatory solutions to quasilinear differential equations. *Georgian Math. J.* **14** (2007), no. 2, 223–238.

[2] F. V. Atkinson, On second-order non-linear oscillations. *Pacific J. Math.* **5** (1955), 643–647.

[3] S. Belohorec, Oscillatory solutions of certain nonlinear differential equations of second order. *Mat.-Fyz. Casopis Sloven. Akad. Vied* **11** (1961), no. 4, 250–255.

[4] I. T. Kiguradze, On the conditions for oscillation of solutions of the differential equation $u'' + a(t)|u|^n \operatorname{sgn}, u = 0$. (Russian) *Časopis Pěst. Mat.* **87** (1962), 492–495.

[5] I. T. Kiguradze, On the oscillatory character of solutions of the equation $d^m u/dt^m + a(t)|u|^n \operatorname{sign} u = 0$. (Russian) *Mat. Sb. (N.S.)* **65(107)** (1964), 172–187.

[6] J. S. W. Wong, Necessary and sufficient conditions for oscillation of second order neutral differential equations. *J. Math. Anal. Appl.* **252** (2000), no. 1, 342–352.