Once More on Typicality and Atypicality of Power-Law Asymptotic Behavior of Solutions to Emden–Fowler Type Differential Equations

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Consider the equation

$$y^{(n)} = p(x, y, y', \dots, y^{(n-1)})|y|^k \operatorname{sign} y, \tag{1}$$

where $n \ge 2$, $k > 1$, and p is a positive continuous function that is Lipschitz-continuous in its last n variables. Also consider a special case of (1), namely,

$$y^{(n)} = p_0|y|^k \operatorname{sign} y \tag{2}$$

with $p_0 > 0$.

Immediate calculations show that equation (2) has positive solutions with exact power-law behavior, namely,

$$y(x) = C(x^* - x)^{-\alpha} \tag{3}$$

defined on $(-\infty,\, x^*)$ with

$$\alpha = \frac{n}{k-1}\,, \quad C = \Big(\frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{p_0}\Big)^{\frac{1}{k-1}}, \tag{4}$$

and arbitrary $x^* \in \mathbb{R}$.

We discuss the problem posed by I. Kiguradze (see [9, Problem 16.4]) on asymptotic behavior of all positive non-extensible (so-called "blow-up") solutions to equations (2) and (1).

For $n = 2$ (see [9]), $n = 3, 4$ (see [1,2], [3, **5.1**]), it appears that if $p(x, y_1, y_2, \dots, y_{n-1})$ tends to p_0 as $x \to x^* - 0$, $y_0 \to \infty, \dots, y_{n-1} \to \infty$, then all such solutions to equation (1) (and equation (2)) have the following power-law asymptotic behavior:

$$y(x) = C(x^* - x)^{-\alpha}(1 + o(1)), \ \ x \to x^* - 0, \tag{5}$$

with α and C defined by (4).

For equation (1) with any n and some additional assumptions on the function p, the existence of solutions with power-law asymptotic behavior is proved, for $5 \le n \le 11$, the existence of an $(n-1)$-parametric family of such solutions is obtained (see [3, **5.1**]).

It is also proved that for weakly super-linear equations (2) (see [5]) and (1) (see [6]) Kiguradze's conjecture on the power-law asymptotic behavior of all blow-up solutions is true.

Theorem 1 ([6])**.** *Suppose that* $p \in C(\mathbb{R}^{n+1}) \cap Lip_{y_0,\dots,y_{n-1}}(\mathbb{R}^n)$ *and* $p \to p_0 > 0$ *as* $x \to x^*$, $y_0 \to \infty, \dots, y_{n-1} \to \infty$. *Then for any integer* $n > 4$ *there exists* $K_n > 1$ *such that for any real* $k \in (1, K_n)$, *any solution to equation* (1) *tending to* $+\infty$ *as* $x \to x^* - 0$ *has the power-law asymptotic behavior* (5).

In the case $n \geq 12$, even if we deal with equation (2), another type of asymptotic behavior of singular solutions appears (see [4, 6, 8, 10]).

Theorem 2 ([8])**.** *For any $n \geq 12$, there exists $k_n > 1$ such that equation* (2) *has a solution $y(x)$ with*

$$y^{(j)}(x) = p_0^{-\frac{1}{k-1}}(x^* - x)^{-\alpha-j} h_j(\log(x^* - x)), \ \ j = 0, 1, \dots, n-1, \tag{6}$$

where all h_j are periodic positive non-constant functions on $\mathbb{R}$.

If we have stronger nonlinearity, then the power-law asymptotic behavior becomes atypical. The following theorem generalizes the results of [7].

Theorem 3. *If $12 \leq n \leq 100000$, then there exists $k_n > 1$ such that at any point $x_0 \in \mathbb{R}$ the set of initial data of asymptotically power-law solutions to equation* (2) *has zero Lebesgue measure whenever $k > k_n$.*

In order to study the blow-up solutions to equation (2) having the vertical asymptote $x = x^*$, we use the substitutions

$$x^* - x = e^{-t}, \quad y = (C + v)e^{\alpha t} \tag{7}$$

with C defined by (4) to transform equation (2) with $p_0 = 1$ to another one, which can be reduced to the first-order system

$$\frac{dV}{dt} = A_\alpha V + F_\alpha(V), \tag{8}$$

where A_α is a constant $n \times n$ matrix with eigenvalues satisfying the equation

$$\prod_{j=0}^{n-1} (\lambda + \alpha + j) = \prod_{j=0}^{n-1} (1 + \alpha + j), \tag{9}$$

and F_α is a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ satisfying

$$\|F_\alpha(V)\| = O(\|V\|^2) \ \text{ and } \ \|F'_{\alpha,V}(V)\| = O(\|V\|) \ \text{ as } \ V \to 0.$$

In order to study equation (1), the same substitution as (7) of variables is used, and a more complicated system than (8) with an additional term $G(t, V)$ appears (see [3]).

The proof of Theorem 3 is based on the following statement.

Lemma. *If there is no purely imaginary root to equation* (9)*, but there exists at least one root not equal to 1 and having positive real part, then for any $x_0 \in \mathbb{R}$, the set of initial data of asymptotically power-law solutions to equation* (2) *has zero Lebesgue measure whenever $k > k_n$.*

Remark. The occurrence of the order 12 for equation (2) in Theorems 2 and 3 is explained by the fact that all roots but one ($\lambda = 1$) to equation (9) with $n < 12$ have negative real parts, which implies the existence of an $(n-1)$-parametric family of solutions with power-law asymptotic behavior (5) of solutions to equation (1). Equation (9) with $n = 12$ and some α has a pair of complex-conjugate purely imaginary roots, which implies the appearance of a solution of the form (6) to equation (2). The order 100000 appearing in Theorem 3 is not final. It is possible to continue the calculations and obtain the same result for equations of order higher than 100000. The previous result (see [7]) was obtained for $12 \leq n \leq 203$.

Open problems

- Is there any blow-up solution to equation (2) with asymptotic behavior other than (5) and (6) ?
- Is there any blow-up solution with non-power-law asymptotic behavior to equation (2) with strong power-law nonlinearity when $5 \le n \le 11$?
- Is it possible to find exactly a constant $K_n^* > 1$ such that for any $k \in (1, K_n^*)$ all blow-up solutions to (2) have power-law asymptotic behavior (5), while other blow-up solutions appear whenever $k > K_n^*$?

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