On Smooth Controllability in Parabolic Control Problem with a Pointwise Observation

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1 Introduction

Consider an extremum problem for the parabolic mixed problem

$$u_t = (a(x)u_x)_x + b(x)u_x + h(x)u, \ \ (x,t) \in Q_T = (0,1) \times (0,T), \ \ T > 0, \tag{1.1}$$

$$u(0,t) = \varphi(t), \quad u_x(1,t) = \psi(t), \ \ 0 < t < T, \tag{1.2}$$

$$u(x,0) = 0, \ \ 0 < x < 1, \tag{1.3}$$

where the real functions a, b and h are smooth in $\overline{Q}_T$, $0 < a_0 \le a(x) \le a_1 < \infty$, $\varphi \in W_2^2(0,T)$, $\psi \in W_2^2(0,T)$. Here $W_2^2(0,T)$ is the Sobolev space of weakly differentiable functions with the norm

$$\|y\|^2_{W_2^k(0,T)} = \int\limits_0^T \Big(\sum_{j=0}^k \big(y^{(k)}(t)\big)^2 \Big)\, dt.$$

We study the control problem with a pointwise observation: by controlling the temperature φ at the left end of the segment (the function ψ is assumed to be fixed), we try to make at some point $x_0 \in (0,1)$ the temperature $u(x_0,t)$ close to the given function $z \in W_2^1(0,T)$ over the entire time interval $(0,T)$. This problem arises in the model of climate control in industrial greenhouses [4,5]. Note that extremal problems for parabolic equations were considered in [11,13–16] (as usual, problems with final or distributed observation). But the results and methods of investigation are not similar to our methods.

Continuing the research in [1–3,6–10], we consider some special quality functional, which is in demand in applications, providing, among other things, uniform proximity of the solution and the objective function, implemented by the norm in the space $W_2^1(0,T)$. Since in applied problems the control and observation time T is sufficiently large, the influence of the initial function is relatively small and can be neglected, setting the initial function equal to zero.

As in [12, p. 6], we denote by $V_2^{1,0}(Q_T)$ Banach space of functions $u \in W_2^{1,0}(Q_T)$ (Sobolev space of functions with the norm $\|u\|^2_{W_2^{1,0}(Q_T)} = \int\limits_{Q_T} (u_x^2 + u^2)\, dx\, dt$) with the finite norm

$$\|u\|_{V_2^{1,0}(Q_T)} = \sup_{0 \le t \le T} \|u(\,\cdot\,, t)\|_{L_2(0,1)} + \|u_x\|_{L_2(Q_T)}$$

such that $t \mapsto u(\,\cdot\,, t)$ is a continuous mapping from $[0, T]$ to $L_2(0, 1)$. Let $\widetilde{W}_2^1(Q_T)$ be the set of all functions $\eta \in W_2^1(Q_T)$, satisfying the conditions $\eta(\,\cdot\,, T) = 0$, $\eta(0, \,\cdot\,) = 0$.

Definition 1.1. A function $u \in V_2^{1,0}(Q_T)$, satisfying the condition $u(0, t) = \varphi(t)$ and the equality

$$\int\limits_{Q_T} \big(a(x)u_x\eta_x - b(x)u_x\eta - h(x)u\eta - u\eta_t\big)\, dx\, dt = a(1)\int\limits_0^T \psi(t)\eta(1, t)\, dt$$

for all $\eta \in \widetilde{W}_2^1(Q_T)$, is called a weak solution to problem (1.1)–(1.3).

2 Main results

Theorem 2.1. *If $\varphi, \psi \in W_2^2(0, T)$ and $\varphi(0) = \psi(0) = 0$, then problem* (1.1)–(1.3) *has a unique weak solution $u \in V_2^{1,0}(Q_T)$ with $u_t \in V_2^{1,0}(Q_T)$, and the inequality*

$$\|u\|_{V_2^{1,0}(Q_T)} + \|u_t\|_{V_2^{1,0}(Q_T)} \le C_1\big(\|\varphi\|_{W_2^2(0,T)} + \|\psi\|_{W_2^2(0,T)}\big) \tag{2.1}$$

holds with some constant C_1, independent of φ and ψ.

Denote by $\Phi \subset W_2^2(0, T)$ nonempty set of control functions φ satisfying the condition $\varphi(0) = 0$, and let $Z \subset W_2^1(0, T)$ be nonempty set of objective functions z satisfying the condition $z(0) = 0$. Consider the functional

$$J[z, \varphi] = \big\|u_\varphi(x_0, t) - z(t)\big\|^2_{W_2^1(0,T)}, \quad \varphi \in \Phi, \quad z \in Z, \tag{2.2}$$

where u_φ is the solution to problem (1.1)–(1.3) with the given control function φ. Considering the function z to be fixed, we have the following minimization problem

$$m[z, \Phi] = \inf_{\varphi \in \Phi} J[z, \varphi]. \tag{2.3}$$

Theorem 2.2. *If the set Φ is closed, convex and bounded in $W_2^2(0, T)$, then for any $z \in Z$ there exists a unique function $\varphi_0 \in \Phi$ such that*

$$m[z, \Phi] = J[z, \varphi_0]. \tag{2.4}$$

Definition 2.1. We will say that problem (1.1)–(1.3), (2.3) is densely controllable from the set Φ to the set Z (see [8, 16]), if for all $z \in Z$ the equality

$$m[z, \Phi] = 0 \tag{2.5}$$

holds.

Theorem 2.3. *Problem* (1.1)–(2.2) *is densely controllable from the set $\Phi = \{\varphi \in W_2^2(0, T) :$ $\varphi(0) = 0\}$ to the set $Z = \{z \in W_2^1(0, T) :\ z(0) = 0\}$.*

3 Proofs

Proof of Theorem 2.1. By results of [8], we can prove that under assumptions of $\varphi \in W_1^2(0,T)$, $\psi \in W_1^2(0,T)$ there exists a unique solution $u \in V_2^{1,0}(Q_T)$ of problem (1.1)–(1.3). This solution satisfies the estimate

$$\|u\|_{V_2^{1,0}(Q_T)} \le C_2\big(\|\varphi\|_{W_2^1(0,T)} + \|\psi\|_{W_2^1(0,T)}\big). \tag{3.1}$$

The function $v = u_t$ is a solution to the problem

$$v_t = (a(x)v_x)_x + b(x)v_x + h(x)v, \ \ (x,t) \in Q_T, \tag{3.2}$$

$$v(0,t) = \varphi'(t), \quad v_x(1,t) = \psi'(t), \ \ 0 < x < 1, \ \ t > 0, \tag{3.3}$$

$$v(x,0) = 0, \ \ 0 < x < 1. \tag{3.4}$$

Using the results of [8], under assumptions of $\varphi' \in W_1^2(0,T)$, $\psi' \in W_1^2(0,T)$ there exists a solution $v \in V_2^{1,0}(Q_T)$ of problem (3.2)–(3.4). This solution satisfies the estimate

$$\|v\|_{V_2^{1,0}(Q_T)} \le C_2\big(\|\varphi'\|_{W_2^1(0,T)} + \|\psi'\|_{W_2^1(0,T)}\big).$$

Therefore,

$$\|u_t\|_{V_2^{1,0}(Q_T)} \le C_2\big(\|\varphi\|_{W_2^2(0,T)} + \|\psi\|_{W_2^2(0,T)}\big). \tag{3.5}$$

Combining estimates (3.1) and (3.5), we obtain the required inequality (2.1). □

The proof of Theorem 2.2 is based on the following lemma concerning the best approximation in Hilbert spaces.

Lemma 3.1 ([4])**.** *Let A be a convex closed set in a Hilbert space H. Then for any $x \in H$ there exists a unique element $y \in A$ such that*

$$\|x - y\| = \inf_{z \in A} \|x - z\|.$$

Proof of Theorem 2.2. Denote

$$B = \big\{y = u_\varphi(x_0, \cdot\,) : \ \ \varphi \in \Phi\big\} \subset W_2^1(0,T).$$

By the convexity of Φ the set B is a convex subset in $W_2^1(0,T)$. The set Φ is bounded and closed in $W_2^1(0,T)$ and by estimate (2.1) we obtain that B is a bounded and closed set in $W_2^1(0,T)$. Now we apply Lemma 3.1 to the case $H = W_2^1(0,T)$, $A = B$, $x = z \in Z \subset H$. By Lemma 3.1 there exists a unique function $y \in B$ such that

$$m[z,\Phi] = \|y - z\|^2_{W_2^1(0,T)}.$$

So, $y = u_{\varphi_0}(x_0, \cdot\,)$ for some $\varphi_0 \in \Phi$ such that

$$m[z,\Phi] = J[z,\varphi_0].$$

Now we can prove that such $\varphi_0 \in \Phi$ is unique by the same technique of maximum principle and unique continuation theorems as in [8]. □

Proof of Theorem 2.3. For $u_\varphi(x_0,0) = z(0) = 0$ we have the representation

$$u_\varphi(x_0,t) - z(t) = \int_0^t (u_{\varphi t}(x_0,\tau) - z'(\tau))\,d\tau, \ \ 0 \le t \le T. \tag{3.6}$$

It follows from (3.6) that

$$\begin{aligned}\big\|u_\varphi(x_0,\,\cdot\,)-z(\,\cdot\,)\big\|^2_{L_2(0,T)}\\ &=\int_0^T\Big(\int_0^t\big(u_{\varphi t}(x_0,\tau)-z'(\tau)\big)\,d\tau\Big)^2\,dt\le\int_0^T t\big\|v_{\varphi'}(x_0,\,\cdot\,)-z'(\,\cdot\,)\big\|^2_{L_2(0,t)}\,dt\\ &\le\frac{T^2}{2}\big\|v_{\varphi'}(x_0,\,\cdot\,)-z'(\,\cdot\,)\big\|^2_{L_2(0,T)}.\end{aligned}\tag{3.7}$$

So, from (3.6) and (3.7) we have

$$\begin{aligned}J[z,\varphi]&=\big\|u_\varphi(x_0,\,\cdot\,)-z(\,\cdot\,)\big\|^2_{W^1_2(0,T)}\\ &=\big\|u_\varphi(x_0,\,\cdot\,)-z(\,\cdot\,)\big\|^2_{L_2(0,T)}+\big\|u_{\varphi t}(x_0,\,\cdot\,)-z'(\,\cdot\,)\big\|^2_{L_2(0,T)}\\ &\le\Big(1+\frac{T^2}{2}\Big)\big\|v_{\varphi'}(x_0,\,\cdot\,)-z'(\,\cdot\,)\big\|^2_{L_2(0,T)}.\end{aligned}\tag{3.8}$$

Now, by the results of [8] and [9], problem (3.2)–(3.4) is densely controllable from $W^1_2(0,T)$ to $L_2(0,T)$. Therefore, for an arbitrary $z'\in L_2(0,T)$ we have

$$\inf_{\varphi'\in W^1_2(0,T)}\big\|v_{\varphi'}(x_0,\,\cdot\,)-z'(\,\cdot\,)\big\|^2_{L_2(0,T)}=0.\tag{3.9}$$

Now, by (3.8), (3.9),

$$\inf_{\varphi\in W^2_2(0,T)}J[z,\varphi]\le\Big(1+\frac{T^2}{2}\Big)\inf_{\varphi'\in W^1_2(0,T)}\big\|v_{\varphi'}(x_0,\,\cdot\,)-z'(\,\cdot\,)\big\|^2_{L_2(0,T)}=0.\qquad\square$$

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