Invariant Toroidal Manifolds of One Class of Discontinuous Dynamical Systems

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1 Introduction

The work [4] is devoted to the study of the main issues of the theory of differential equations with impulse action. In [2], conditions are established that guarantee the hyperbolicity of systems of differential equations with impulse action. The obtained hyperbolicity conditions allow us to study the existence of bounded solutions of inhomogeneous multidimensional systems of differential equations with impulse perturbation. In [1], sufficient conditions for the existence of an asymptotically stable invariant toroidal manifold of linear extensions of a dynamical system on a torus are obtained in the case of a matrix of the system that commutes with its integral. The proposed approach is applied to the study of the stability of invariant sets of a certain class of discontinuous dynamical systems. In [3], a review of the most modern methods for studying the stability of solutions of impulse differential equations and their application to impulse control problems is carried out. The exponential stability of a trivial torus is proved for one class of nonlinear extensions of dynamical systems on a torus. The obtained results are applied to the study of the stability of toroidal sets of impulsive dynamical systems. The concept of an impulsive non-autonomous dynamical system is introduced. For it, the existence and properties of an impulsive attracting set are investigated. The obtained results are applied to the study of the stability of a two-dimensional impulsive-perturbed Navier–Stokes system. In all the above works, the foundations of the qualitative theory of differential equations with impulsive action are outlined. In essence, the foundations of the qualitative theory of impulsive systems were laid, which are based on the qualitative theory of differential equations, methods of asymptotic integration of such equations, the theory of difference equations and generalized functions. At the same time, the issues of the existence of solutions of weakly nonlinear impulsive systems have not yet been fully investigated.

2 Setting of the problem and the main results

Consider a system of differential equations that is subject to impulse action when an angular variable passes through ϕ fixed set Γ on a torus $\mathfrak{T}^m$:

$$\begin{cases} \dfrac{d\phi}{dt} = \omega, \quad \dfrac{dx}{dt} = A(\phi)x + f(\phi, x), \quad \phi \notin \Gamma, \\ \triangle x\big|_{\phi\in\Gamma} = B(\phi)x + I(\phi, x), \quad \triangle x\big|_{\phi\in\Gamma} = a, \end{cases} \tag{2.1}$$

where $x \in R^n$, $\phi \in \mathfrak{T}^m$ and a – constant m-dimensional vectors, $\Gamma(m-1)$ is a dimensional manifold defined by the equation $\langle k, \phi\rangle = 0 \pmod{2\pi}$ and $k = (k_1, \dots, k_m)$ is an integer vector such that $\langle k, \phi\rangle = 0 \pmod{2\pi}$.

The last condition ensures that the angular phase variable belongs to the $\phi(t)$ set Γ at the moment of the pulse action both at $t - 0$, and at $t + 0$. $A(\phi)$ and $B(\phi)$ are continuous, 2π are square matrices that are periodic in each component ϕ_i, the functions $f(\phi, x)$ and $I(\phi, x)$, defined for all $\phi \in \mathfrak{T}^m$, are continuous (piecewise continuous with discontinuities of the first kind in ϕ), 2π-periodic in ϕ and satisfy the Lipschitz condition uniformly with respect to $\phi \in \mathfrak{T}^m$:

$$\left\| f(\phi, x') - f(\phi, x'') \right\| + \left\| I(\phi, x') - I(\phi, x'') \right\| \leq N \|x' - x''\| \tag{2.2}$$

for each $x' \in R^n$.

Let us denote by the $\phi_t(\phi)$ solution of the system

$$\frac{d\phi}{dt} = \omega, \quad \phi \not\subseteq \Gamma, \quad \triangle x\big|_{\phi \in \Gamma} = a. \tag{2.3}$$

Let us find out under what condition the trajectory of this solution densely fills the torus everywhere $\mathfrak{T}^m$. Any continuous trajectory $\phi = \omega t + \phi_0$ crosses the manifold Γ at equal time intervals $\beta = \frac{2\pi}{\langle k,\omega\rangle}$.

Consider the trajectory of system (2.3) passing through the point $\phi_t = \phi_t(0)$. The next point of intersection of it with Γ will be $\frac{2\pi}{\langle k,\omega\rangle}\omega$. After the first jump we get a point $\frac{2\pi}{\langle k,\omega\rangle}\omega + a$.

Let us call a section of motion consisting of one continuous arc and one jump one stroke of motion. The starting points of individual strokes of motion are, as we see, the points $S(\frac{2\pi}{\langle k,\omega\rangle}\omega + a)$ where S is an integer.

We obtain the following statement.

Lemma 2.1. *Any continuous trajectory $\phi_t(\phi)$ of system (2.3) is closed if and only if the coordinates of the vector $D = \frac{1}{\langle k,\omega\rangle}\,\omega + \frac{1}{2\pi}\,a$ are rational.*

Before formulating the main result related to system (2.1), consider the following system of equations:

$$\begin{cases} \dfrac{d\phi}{dt} = \omega, \quad \dfrac{dx}{dt} = A(\phi)x + f(\phi), \quad \phi \not\subseteq \Gamma, \\ \triangle x\big|_{\phi \in \Gamma} = B(\phi)x + I(\phi), \quad \triangle x\big|_{\phi \in \Gamma} = a \end{cases} \tag{2.4}$$

in which $A(\phi)$, $B(\phi)$, ω, a, Γ – the same as in system (2.1); $f(\phi)$ and $I(\phi)$ – continuous (piecewise continuous with discontinuities of the first kind along ϕ), 2π-periodic in ϕ function.

Let us denote by $I(t, \tau, \phi)$ the normalized fundamental matrix of the system

$$\begin{cases} \dfrac{dx}{dt} = A(\phi_t(\phi))x, \quad t \neq t_i(\phi), \\ \triangle x\big|_{t=t_i} = B(\phi_t(\phi)). \end{cases} \tag{2.5}$$

Note that, as shown in [4], $I(t, \tau, \phi)$ the system of differential equations $\frac{dx}{dt} = A(\phi_t(\phi))x$ associated with the matricant $\Omega^t_\tau(\phi)$ as follows:

$$I(t, \tau, \phi) = \Omega^t_{t_i(\phi)} \Pi \Big[(E + B_{i-k+1}) \Omega^{t_{i-k+1}(\phi)}_{t_{i-k}(\phi)} \Big], \ \ t_i(\phi) \leq t \leq t_{i+1}(\phi).$$

In the following, we assume that $I(t, \tau, \phi)$ satisfies the inequality

$$|I(t, \tau, \phi)| \leq K e^{-\gamma(t-\tau)}, \ \ t \geq \tau. \tag{2.6}$$

Lemma 2.2. *Let in the system of equations (2.4) the functions $f(x)$ and $I(\phi)$ be periodic, continuous (piecewise continuous on τ^m). $A(\phi)$ and $B(\phi)$ are continuous on τ^m 2π- periodic matrices. If the matrix $I(t, \tau, \phi)$ satisfies estimate (2.6), then the system of equations (2.4) has an asymptotically stable invariant set*

$$x = u(\phi), \quad u(\phi + 2\pi) = u(\phi),$$

where $u(\phi)$ is a piecewise-continuous function with discontinuities of the first kind on the set Γ such that for some constant C we obtain the inequality:

$$\|U(\phi)\| \leq C \cdot \max\Big\{ \max_{\phi \in \tau^m} \|f(\phi)\|, \max_{\phi \in \tau^m} \|I(\phi)\| \Big\}.$$

Thus, the function $u(\phi)$ defines the invariant set of the system of equations (2.4). The asymptotic stability of this set is ensured by inequality (2.6).

Let us note some special cases of systems (2.5) for which the fundamental matrix $Y(t,\tau,\phi)$ implies estimate (2.6).

Inequality (2.6) is also satisfied if A and B are constant matrices that commute with each other, and are non-degenerate and the real parts of all eigenvalues $E+B$ of the matrix $\wedge = A + \frac{\langle k,\omega\rangle}{2\pi} \ln(E+B)$ are negative.

So, based on the results obtained, we obtain the following theorem.

Theorem. *Let the system of equations* (2.1) *be such that inequalities* (2.2) *and* (2.6) *are satisfied. Then we can specify a positive number* N_0 *such that for all* $0 \leq N \leq N_0$ *the system of equations* (2.1) *has an asymptotically stable invariant set* $x = u(\phi), u(\phi + 2\pi) = u(\phi)$*, where* $u(\phi)$ *is a piecewise-continuous function with discontinuities of the first kind on the set* Γ *such that*

$$\triangle U\big|_{\phi \in \Gamma} = B(\phi)u(\phi) + I(\phi, u(\phi)).$$

References

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