Parameter-Dependent Periodic Problems for Non-Autonomous Duffing Equations with a Sign-Changing Forcing Term

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The extended abstract concerns the parameter-dependent periodic problem

$$u'' = p(t)u + h(t)|u|^\lambda \operatorname{sgn} u + \mu f(t); \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega), \tag{1}$$

where $p, h, f \in L([0,\omega])$, $h \geq 0$ a.e. on $[0,\omega]$, $\lambda > 1$, and $\mu \in \mathbb{R}$ is a parameter. By a solution to problem (1), as usual, we understand a function $u : [0,\omega] \to \mathbb{R}$ which is absolutely continuous together with its first derivative, satisfies the given equation almost everywhere, and meets the periodic conditions. The text is based on the paper [4].

We first show where problem (1) may appear from. Consider a forced oscillator consisting of two fixed charged bodies of charges $q > 0$ and a charged mass body of weight m and charge $Q > 0$ (see Fig. 1).

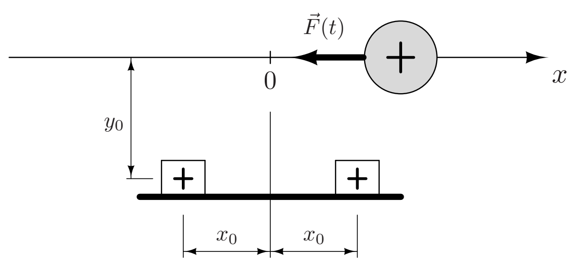

Figure 1. Nonlinear undamped forced oscillator.

Assume that the mass body moves horizontally without any friction and the charges q of the fixed bodies change ω-periodically, i.e., $q : \mathbb{R} \to]0,+\infty[$ is an ω-periodic function. This is a system with one degree of freedom described by the coordinate x, whose equation of motion is of the form

$$mx'' - \frac{Qq(t)}{4\pi\varepsilon_r\varepsilon_0}\left(\frac{x+x_0}{[(x+x_0)^2+y_0^2]^{3/2}} + \frac{x-x_0}{[(x-x_0)^2+y_0^2]^{3/2}}\right) = F(t), \tag{2}$$

where ε_r is the relative permittivity and ε_0 is the vacuum permittivity.

Numeric simulations show that if $y_0^2 < 2x_0^2$, then equation (2) with $q(t) \equiv Const.$ and $F(t) \equiv 0$ has exactly three equilibria $x_1 := 0$, $x_2 > 0$, and $x_3 = -x_2$. Approximating the non-linearity in (2) by the third degree Taylor polynomial centred at 0, we obtain the equation

$$x'' = -\frac{Qq(t)(2x_0^2 - y_0^2)}{2\pi\varepsilon_r\varepsilon_0 m(x_0^2+y_0^2)^{5/2}}\,x + \frac{3Qq(t)(24x_0^2y_0^2 - 3y_0^4 - 8x_0^4)}{\pi\varepsilon_r\varepsilon_0 m(x_0^2+y_0^2)^{9/2}}\,x^3 + \frac{F(t)}{m}\,,$$

which is a particular case of the differential equation in (1) with $\mu = 1$, where

$$p(t) := -\frac{Qq(t)(2x_0^2 - y_0^2)}{2\pi\varepsilon_r\varepsilon_0 m(x_0^2 + y_0^2)^{5/2}}, \quad h(t) := \frac{3Qq(t)(24x_0^2y_0^2 - 3y_0^4 - 8x_0^4)}{\pi\varepsilon_r\varepsilon_0 m(x_0^2 + y_0^2)^{9/2}},$$

$f(t) := \frac{F(t)}{m}$, and $\lambda := 3$. Assuming that $(4 - 2\sqrt{10/3}\,)x_0^2 < y_0^2 < 2x_0^2$ and $F(t) \not\equiv 0$, it is easy to show that the functions p and h are negative and positive, respectively.

To formulate our results, we need the following definition.

Definition ([2])**.** We say that a function p belongs to the set $\mathcal{V}^-(\omega)$ (resp. $\mathcal{V}^+(\omega)$) if, for any function $u \in AC^1([0,\omega])$ satisfying

$$u''(t) \geq p(t)u(t) \quad \text{for a. e. } t \in [0,\omega], \quad u(0) = u(\omega), \;\; u'(0) \geq u'(\omega),$$

the inequality $u(t) \leq 0$ (resp. $u(t) \geq 0$) holds for $t \in [0,\omega]$. By $\mathcal{U}(\omega)$, we denote the set of pairs (p,f) such that the problem

$$u'' = p(t)u + f(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega) \tag{3}$$

has a unique solution which is positive. The set $\mathcal{V}_0(\omega)$ consists of all the functions p such that problem (3) with $f(t) \equiv 0$ possesses a positive solution.

Remark 1. The effective conditions guaranteeing the inclusions $p \in \mathcal{V}^-(\omega)$, $p \in \mathcal{V}^+(\omega)$, $p \in \mathcal{V}_0(\omega)$, as well as $(p,f) \in \mathcal{U}(\omega)$ are provided in [2] (see also [1,5]).

Below we discuss the existence/non-existence as well as the exact multiplicity of positive solutions to problem (1) depending on the choice of the parameter μ provided that $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$. Let us show, as a motivation, what happens in the autonomous case of (1). Hence, we consider the equation

$$x'' = -ax + b|x|^\lambda \operatorname{sgn} x + \mu. \tag{4}$$

In view of our hypotheses $h \geq 0$ a. e. on $[0,\omega]$, $h(t) \not\equiv 0$ and since $-a \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ if only if $a > 0$, we assume that $a, b > 0$. By direct calculation, the phase portraits of equation (4) can be elaborated depending on the choice of the parameter $\mu \in \mathbb{R}$ (see, Fig. 2) and, thus, one can prove the following proposition concerning the periodic solutions to equation (4).

Proposition 1. *Let $\lambda > 1$ and $a, b > 0$. Then, the following conclusions hold:*

- (i) *If $\mu > \frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, then equation* (4) *has a unique negative equilibrium (saddle) and no other periodic solutions occur.*
- (ii) *If $\mu = \frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, then equation* (4) *has a unique positive equilibrium (cusp), a unique negative equilibrium (saddle), and no other periodic solutions occur.*
- (iii) *If $0 < \mu < \frac{(\lambda-1)a}{\lambda}\left(\frac{a}{\lambda b}\right)^{\frac{1}{\lambda-1}}$, then equation* (4) *possesses exactly two positive equilibria $x_1 > x_2$ (x_1 is a saddle and x_2 is a center), a unique negative equilibrium x_3 (saddle), and non-constant (positive and possibly sign-changing) periodic solutions with different periods. Moreover, all the non-constant periodic solutions oscillate around x_2 between x_3 and x_1.*
- (iv) *If $\mu = 0$, then equation* (4) *possesses a unique positive equilibrium x_0 (saddle), a trivial equilibrium (center), a unique negative equilibrium $-x_0$, and non-constant sign-changing periodic solutions with different periods. Moreover, all the non-constant periodic solutions oscillate around* 0 *between $-x_0$ and x_0.*
- (v) *For $\mu < 0$, the conclusions are "symmetric" as compared with the items* (i)–(iii)*, see Fig. 2.*

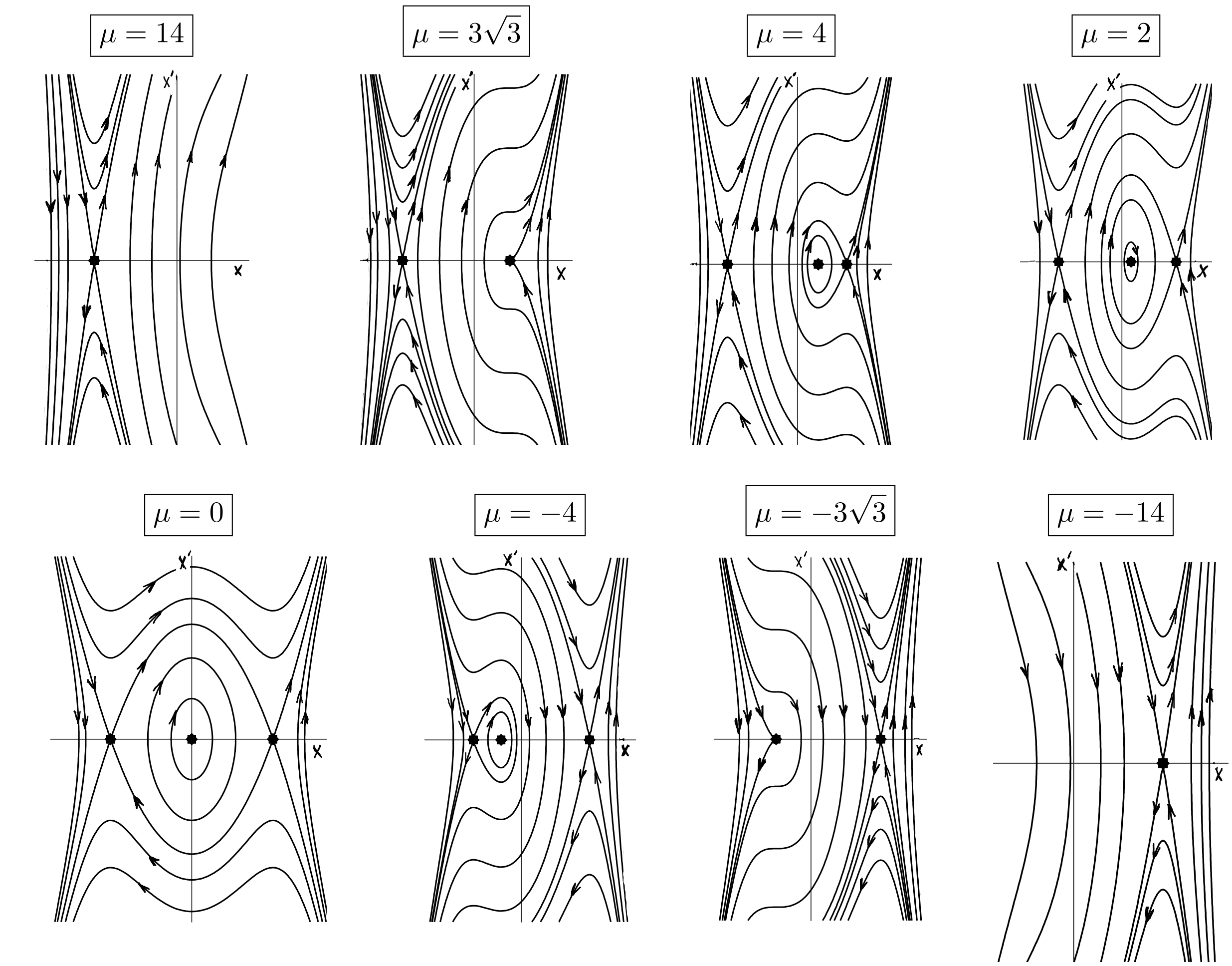

Figure 2. Phase portraits of equation (4) with $a = 9$, $b = 4$, and $\lambda = 3$.

We start with the most general statement of the text, which provides the existence/non-existence results in the case of $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$. This condition is satisfied, e.g., if $\int\limits_0^\omega p(s)\mathrm{d}s \leq 0$, $p(t) \not\equiv 0$.

Theorem 1. *Let $\lambda > 1$, $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, $f(t) \not\equiv 0$, and*

$$h(t) > 0 \quad \text{for a. e. } t \in [0, \omega]. \tag{5}$$

Then, there exist $-\infty \leq \mu_* < 0$ and $0 < \mu^* \leq +\infty$ such that the following conclusions hold:

(1) *For any $\mu \in]\mu_*, \mu^*[$, problem (1) has a positive solution u^* such that every solution u to problem (1) satisfies*

$$\text{either } u(t) < u^*(t) \quad \text{for } t \in [0, \omega], \quad \text{or } u(t) \equiv u^*(t). \tag{6}$$

Moreover, any couple of distinct positive solutions u_1, u_2 to (1) different from u^* satisfies

$$\min\big\{u_1(t) - u_2(t) : \ t \in [0, \omega]\big\} < 0, \quad \max\big\{u_1(t) - u_2(t) : \ t \in [0, \omega]\big\} > 0.$$

(2) *If $\mu^* < +\infty$ (e.g. provided that $\int\limits_0^\omega f(s)\mathrm{d}s > 0$), then*

(a) *for* $\mu > \mu^*$, *problem* (1) *has no positive solution,*

(b) *for* $\mu = \mu^*$, *problem* (1) *has a unique non-negative solution* u^* *and every solution* u *to* (1) *satisfies* (6).

(3) *If* $\mu_* > -\infty$ *(e.g. provided that* $\int\limits_0^\omega f(s)\mathrm{d}s < 0$*), then*

(a) *for* $\mu < \mu_*$, *problem* (1) *has no positive solution,*

(b) *for* $\mu = \mu_*$, *problem* (1) *has a unique non-negative solution* u^* *and every solution* u *to* (1) *satisfies* (6).

It is clear that u is a solution to problem (1) if and only if $-u$ is a solution to the problem

$$z'' = p(t)z + h(t)|z|^\lambda \operatorname{sgn} z - \mu f(t); \quad z(0) = z(\omega), \ \ z'(0) = z'(\omega).$$

Therefore, we get the following corollary from Theorem 1.

Corollary. *Let* $\lambda > 1$, $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, $f(t) \not\equiv 0$, *and condition* (5) *hold. Then, there exists* $0 < \mu_0 < +\infty$ *such that, for any* $\mu \in]-\mu_0, \mu_0[$, *problem* (1) *has a negative solution* u_* *and a positive solution* u^* *such that every solution* u *to problem* (1) *different from* u_*, u^* *satisfies*

$$u_*(t) < u(t) < u^*(t) \quad \textit{for} \ \ t \in [0, \omega].$$

We showed in [3, Example 2.8] that assuming $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, hypothesis (5) in Theorem 1 (i.e. the positivity of h a. e. on $[0, \omega]$) is essential for the existence of a positive solution to problem (1) with $\mu = 0$ and cannot be weakened to the non-negativity of h. However, under a stronger assumption on the coefficient p (namely, $p \in \mathcal{V}^+(\omega)$), hypothesis (5) of Theorem 1 can be relaxed to

$$h(t) \geq 0 \quad \text{for a. e. } \ t \in [0, \omega], \ \ h(t) \not\equiv 0. \tag{7}$$

Theorem 2. *Let* $\lambda > 1$, $p \in \mathcal{V}^+(\omega)$, h *satisfy* (7), *and*

$$(p, f) \in \mathcal{U}(\omega), \qquad \int\limits_0^\omega f(s)\mathrm{d}s > 0. \tag{8}$$

Then, there exist $-\infty \leq \mu_* < 0$ *and* $0 < \mu^* < +\infty$ *such that the following conclusions hold:*

(1) *For any* $\mu > \mu^*$, *problem* (1) *has no positive solution.*

(2) *For* $\mu = \mu^*$, *problem* (1) *has a unique positive solution* u^* *and, moreover, every solution* u *to problem* (1) *satisfies* (6).

(3) *For* $\mu \in]0, \mu^*[$, *problem* (1) *has exactly two positive solutions* u_1, u_2 *and these solutions satisfy*

$$u_1(t) > u_2(t) > 0 \quad \textit{for} \ \ t \in [0, \omega].$$

Moreover, every solution u *to problem* (1) *different from* u_1 *is such that*

$$u(t) < u_1(t) \quad \textit{for} \ \ t \in [0, \omega].$$

(4) *For* $\mu = 0$, *problem* (1) *has exactly three solutions: a positive solution* u_0, *the trivial solution, a negative solution* $-u_0$.

(5) *For* $\mu \in]\mu_*, 0[$, *problem* (1) *has either one or two positive solutions. Moreover,* (1) *has a positive solution* u^* *such that every solution* u *to problem* (1) *satisfies* (6).

(6) *If $\mu_* > -\infty$, then, for any $\mu < \mu_*$, problem* (1) *has no positive solution.*

Open questions. The following two questions remain open in Theorem 2:

(a) Does the inequality $\mu_* > -\infty$ hold without any additional assumption?
(b) What happens in the case of $\mu = \mu_*$, if $\mu_* > -\infty$ and $h(t) = 0$ on a set of positive measure?

Remark 2. Assuming $f(t) \geq 0$ for a. e. $t \in [0, \omega]$, $f(t) \not\equiv 0$, the conclusions of Theorems 1 and 2 can be substantially refined (see [4, Theorems 3.6 and 3.14]).

Theorem 2 guarantees the existence of certain "critical" values μ_*, μ^* of the parameter μ such that crossing these values, a bifurcation of positive solutions to problem (1) occurs. From an application point of view, the estimates of these numbers are also needed.

Proposition 2. *Let $\lambda > 1$, $p \in \operatorname{Int} \mathcal{V}^+(\omega)$, h satisfy* (7)*, and*

$$\int_0^\omega [f(s)]_+ \mathrm{d}s > \Gamma(p) \int_0^\omega [f(s)]_- \mathrm{d}s > 0,$$

where the number $\Gamma(p)$, depending only on p, is defined in [2, Section 6]. Then, the numbers μ_*, μ^* appearing in the conclusion of Theorem 2 satisfy

$$\mu_* \leq -\frac{(\lambda - 1)[\Delta(p)]^{-\frac{\lambda}{\lambda-1}}}{\lambda\big[\lambda \int_0^\omega h(s)\mathrm{d}s\big]^{\frac{1}{\lambda-1}} \int_0^\omega [f(s)]_- \mathrm{d}s},$$

where $\Delta(p)$ denotes a norm of Green's operator of problem (8) *(see [4, Remark 2.5]), and*

$$\frac{(\lambda - 1)[\Delta(p)]^{-\frac{\lambda}{\lambda-1}}}{\lambda\big[\lambda \int_0^\omega h(s)\mathrm{d}s\big]^{\frac{1}{\lambda-1}} \int_0^\omega [f(s)]_+ \mathrm{d}s} \leq \mu^* < \frac{(\lambda - 1)\big[\Gamma(p) \int_0^\omega [p(s)]_- \mathrm{d}s - \int_0^\omega [p(s)]_+ \mathrm{d}s\big]^{\frac{\lambda}{\lambda-1}}}{\lambda\big[\lambda \int_0^\omega h(s)\mathrm{d}s\big]^{\frac{1}{\lambda-1}} \big[\int_0^\omega [f(s)]_+ \mathrm{d}s - \Gamma(p) \int_0^\omega [f(s)]_- \mathrm{d}s\big]}.$$

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