Stability and Exponential Stability Indices of a Linear System Depending on a Parameter

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1 Introduction

For a given integer $n \geq 2$ let $\mathcal{M}_n$ denote the class of linear differential systems

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}_+ \equiv [0, +\infty), \tag{1.1}$$

with continuous bounded coefficients defined on $\mathbb{R}_+$.

Let us identify the system (1.1) with the matrix-valued function $A : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ and use the following notation: $A \in \mathcal{M}_n$.

Recall that the *characteristic exponent* (or the *Lyapunov exponent*) of a function $x(\,\cdot\,)$ is [6, p. 552], [1, p. 25]

$$\chi(x) = \varlimsup_{t\to+\infty} \ln \|x(t)\|^{1/t}.$$

We denote by $\mathrm{s}(A)$ the stability index of a system $A \in \mathcal{M}_n$, i.e. the dimension of the linear subspace of bounded solutions to this system, and by $\mathrm{es}(A)$ its exponential stability index, that is the dimension of the linear subspace of solutions to this system having negative characteristic exponents. One can see that the following inequality holds:

$$\mathrm{s}(A) \geq \mathrm{es}(A).$$

O. Perron in his paper [7] constructed an example of a system $A \in \mathcal{M}_2$ and its continuous exponentially decaying (2×2)-dimensioned perturbation $Q(\,\cdot\,)$ such that the initial system has its exponential stability index equal to 2, and the perturbed system

$$\dot{x} = (A(t) + Q(t))x, \ \ x \in \mathbb{R}^2, \ \ t \in \mathbb{R}_+,$$

has both its stability indices equal to 1.

There is another interesting example presented by O. Perron [8]. He describes a diagonal system $A \in \mathcal{M}_2$ with the exponential stability index equal to 2 (and with the same stability index) and its continuous higher-order perturbation $f(\,\cdot\,,\,\cdot\,)$ such that the stability index of the perturbed system $\dot{x} = A(t)x + f(t, x)$ equals 0 (and so does its exponential stability index).

Thus, both examples demonstrate the effect of loss of stability. These examples had initiated a lot of research aimed to learn how perturbations of different types can affect stability of systems in $\mathcal{M}_n$. The results obtained in this direction form a considerable part of the contemporary Lyapunov exponent theory. When a perturbation is in a certain sense "small", the effect of loss of stability is called the Perron effect [5, Chapter 4]. Starting with the paper [4], this term is used only when perturbations do not decrease the Lyapunov exponents of the initial system, and we adhere to this terminology.

2 Statement of the problem and the main result

In this paper we present a kind of generalization of the Perron effect. To this end, for a system $A \in \mathcal{M}_n$ and a metric space M we consider the class $\mathcal{E}_n[A](M)$ consisting of jointly continuous matrix-valued functions $Q : \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$ satisfying the following two conditions.

The first one is the estimate

$$\|Q(t,\mu)\| \le C_Q \exp(-\sigma_Q t) \ \text{ for all } \ (t,\mu) \in \mathbb{R}_+ \times M,$$

where C_Q and σ_Q are positive constants (generally, different for each function Q).

The second condition is that the stability and exponential stability indices of the perturbed system $A + Q$, being functions of $\mu \in M$ and denoted by $\mathrm{s}(\,\cdot\,; A+Q)$ and $\mathrm{es}(\,\cdot\,; A+Q)$, do not exceed the corresponding stability indices of the system A, i.e.

$$\mathrm{s}(\mu; A+Q) \le \mathrm{s}(A) \ \text{ and } \ \mathrm{es}(\mu; A+Q) \le \mathrm{es}(A) \ \text{ for all } \ \mu \in M.$$

We state the problem in the following way. Our task is for any integer $n \ge 2$ and metric space M to give a complete functional description of the class of pairs $((\mathrm{s}(A), \mathrm{es}(A)), (\mathrm{s}(\,\cdot\,; A+Q), \mathrm{es}(\,\cdot\,; A+Q)))$ composed of the stability indices of the initial system A and those of the perturbed system $A+Q$. The system A here ranges over $\mathcal{M}_n$, and for every A the matrix-valued function Q ranges over the set $\mathcal{E}_n[A](M)$. Thus, our problem is to present a complete functional description of the following class:

$$\Sigma\,\mathcal{E}_n(M) \equiv \Big\{ \big((\mathrm{s}(A), \mathrm{es}(A)), (\mathrm{s}(\,\cdot\,; A+Q), \mathrm{es}(\,\cdot\,; A+Q))\big) \mid \ A \in \mathcal{M}_n, \ \ Q \in \mathcal{E}_n[A](M) \Big\}.$$

Before we could formulate the main result, let us remind the reader that a function $f : M \to \mathbb{R}$ is called [3, pp. 266–267] a function of the class $(F_\sigma, {}^*)$ if for any $r \in \mathbb{R}$ the preimage of the half-line $(r, +\infty)$ is an F_σ-set in the space M, i.e. it can be represented as a countable union of closed subsets of M. In particular, the class $(F_\sigma, {}^*)$ is a subclass of Baire class 2 [3, p. 294]. Let us also denote the set $\{0, 1, \ldots, n\}$ by $\mathcal{Z}_n$.

The solution to the problem is stated by the following

Theorem 1. *Let M be a metric space and $n \geqslant 2$ an integer. A pair $((\alpha_0, \beta_0), (\alpha(\,\cdot\,), \beta(\,\cdot\,))$ with $\alpha_0, \beta_0 \in \mathcal{Z}_n$ and $\alpha(\,\cdot\,), \beta(\,\cdot\,) : M \to \mathcal{Z}_n$ belongs to the class $\Sigma\,\mathcal{E}_n(M)$ if and only if the following conditions are met:*

1) $\alpha_0 \ge \beta_0$*;*

2) $\alpha(\mu) \ge \beta(\mu)$ *for all* $\mu \in M$*;*

3) $\alpha(\mu) \le \alpha_0$,$\beta(\mu) \le \beta_0$ *for all* $\mu \in M$*;*

4) *the functions* $\alpha(\,\cdot\,)$ *and* $\beta(\,\cdot\,)$ *are of the class* $(F_\sigma, {}^*)$*.*

3 Corollaries and remarks

Let M be a metric space. For an integer $n \ge 2$ we consider a family of linear systems depending on a parameter $\mu \in M$ of the form

$$\dot{x} = \mathcal{A}(t,\mu)x, \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}_+, \tag{3.1}$$

such that for each fixed $\mu \in M$ the matrix-valued function $\mathcal{A}(\,\cdot\,,\mu) : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is continuous and bounded (generally, the bounding constant is different for each μ).

Usually, a family of mappings $\mathcal{A}(\,\cdot\,,\mu)$, $\mu \in M$, is considered under one of the two following natural assumptions: the family is continuous in **a)** the compact-open topology or **b)** the uniform topology. The case **a)** is equivalent to the condition that if a sequence $(\mu_k)_{k\in\mathbb{N}}$ of points from M converges to a point $\mu_0 \in M$, then the sequence $(\mathcal{A}(t,\mu_k))_{k\in\mathbb{N}}$ of matrices converges uniformly over each interval $[0,T] \subset \mathbb{R}_+$ to the matrix $\mathcal{A}(t,\mu_0)$ as $k \to +\infty$. The case **b)** differs from **a)** in that the convergence is uniform over the whole half-line $\mathbb{R}_+$. We denote the class of families (3.1) that are continuous in the compact-open topology by $\mathcal{C}^n(M)$ and those that are continuous in the uniform topology by $\mathcal{U}^n(M)$. Clearly, $\mathcal{U}^n(M) \subset \mathcal{C}^n(M)$.

Further, we identify the family (3.1) with the matrix-valued function $\mathcal{A}(\,\cdot\,,\,\cdot\,)$ specifying it and use the following notation: $\mathcal{A} \in \mathcal{C}^n(M)$ or $\mathcal{A} \in \mathcal{U}^n(M)$.

Corollary. *Let M be a metric space. For any integer $n \geq 2$, the classes of pairs of functions*

$$\Sigma\,\mathcal{C}_n(M) \equiv \big\{(\mathrm{s}(\,\cdot\,;\mathcal{A}), \mathrm{es}(\,\cdot\,;\mathcal{A})) \mid \ \mathcal{A} \in \mathcal{C}^n(M)\big\}$$

and

$$\Sigma\,\mathcal{U}_n(M) \equiv \big\{(\mathrm{s}(\,\cdot\,;\mathcal{A}), \mathrm{es}(\,\cdot\,;\mathcal{A})) \mid \ \mathcal{A} \in \mathcal{U}^n(M)\big\}$$

coincide with one another and consist of the pairs $(\alpha(\,\cdot\,),\beta(\,\cdot\,))$ of functions $M \to \mathcal{Z}_n$ of the class $(F_\sigma, {}^*)$ that satisfy the inequality $\alpha(\mu) \geq \beta(\mu)$ for all $\mu \in M$.

Remark. The description of the classes composed of the second elements of the pairs from $\Sigma\,\mathcal{C}_n(M)$ and $\Sigma\,\mathcal{U}_n(M)$ was obtained in the paper [2]. Those classes coincide with one another and consist of functions $M \to \mathcal{Z}_n$ from the class $(F_\sigma, {}^*)$.

References

[1] L. Ya. Adrianova, *Introduction to Linear Systems of Differential Equations.* Translated from the Russian by Peter Zhevandrov. Translations of Mathematical Monographs, 146. American Mathematical Society, Providence, RI, 1995.

[2] E. A. Barabanov, V. V. Bykov and M. V. Karpuk, Complete description of the exponential stability index for linear parametric systems as a function of the parameter. (Russian) *Differ. Uravn.* **55** (2019), no. 10, 1307–1318; translation in *Differ. Equ.* 55 (2019), no. 10, 1263–1274.

[3] F. Hausdorff, *Set Theory.* Second edition. Translated from the German by John R. Aumann et al. Chelsea Publishing Co., New York, 1962.

[4] S. K. Korovin and N. A. Izobov, Realization of the Perron effect for the change of values of the characteristic exponents of solutions of differential systems. (Russian) *Differ. Uravn.* **46** (2010), no. 11, 1536–1550; translation in *Differ. Equ.* **46** (2010), no. 11, 1537–1551.

[5] G. A. Leonov, *Chaotic Dynamics and Classical Theory of Motion Stability.* (Russian) Reguljarnaja i haoticheskaja dinamika, Izhevsk, 2006.

[6] A. M. Lyapunov, The general problem of the stability of motion. *Int. J. Control* **55** (1992), no. 3, 531–773.

[7] O. Perron, Die Ordnungszahlen linearer Differentialgleichungssysteme. (German) *Math. Z.* **31** (1930), no. 1, 748–766.

[8] O. Perron, Die Stabilitätsfrage bei Differentialgleichungen. (German) *Math. Z.* **32** (1930), no. 1, 703–728.