General Theory of the Higher-Order Linear Quaternion q-Difference Equations

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Abstract

The general theory of quaternion q-difference equations is essentially different from the traditional q-difference equations in the complex space for the non-commutative algebraic structure of the quaternionic algebra. In this talk, we will present some basic results such as the Wronskian, Liouville formula and the general solution structure theorems of the higher-order linear quaternion q-difference equations (short for QQDCEs) with constant and variable coefficients by applying the quaternion characteristic polynomial and the quaternion determinant algorithm.

1 Introduction

In 1843, Hamilton initiated the quaternion space $\mathbb{H}$ to extend and develop the complex field $\mathbb{C}$ and applied it to mechanics in three-dimensional space. With the large development of the quaternion algebra, it demonstrates a great superiority over the real-valued vectors and has been widely applied to depicting some complex phenomena in physics, space geometric analysis, especially in the aspects of flight dynamics, molecular dynamics and three-dimensional rotations, etc. (see [1, 2]).

The unified form called dynamic equations on time scales were introduced to combine these both continuous and discrete forms and the common features of the continuous and discrete dynamic equations have been extensively studied (see [5]). Recently, the quaternion differential and difference equations have been widely studied in both theoretical aspects and application area, the quaternionic dynamics described by these equations perfectly present the dynamical behavior of the status of the objects comparing with the complex equations since the various shift transforms such as the rotations in the quaternion space can be easily expressed and accurately calculated. In 2020–2021, the authors established some basic results of quaternion dynamic equations on time scales, and some real applications were provided (see [3, 4]).

In [6], Wang, Chen and Li established the general theory of the higher-order quaternion linear difference equations via the complex adjoint matrix and the quaternion characteristic polynomial and it is largely different from the general theory of the traditional difference equations since the non-commutativity under the quaternion multiplication (i.e., $ab \neq ba$ for $a, b \in \mathbb{H}$). In [7], the general theory of the higher-order linear quaternion q-difference equations was established.

2 Preliminaries

We assume that $0 < q < 1$, $\overline{q^{\mathbb{N}}} := \{q^n : n \in \mathbb{N}\} \cup \{0\}$. For convenience, we introduce some notations. The symbol $\mathbb{N}$ denotes the set of non-negative integers, $\mathbb{C}$ the complex numbers, $M_m(\mathbb{H})$ the $m \times m$-order quaternionic matrices and $M_m(\mathbb{C})$ the $m \times m$-order complex matrices.

Next, some basic knowledge of the quaternion algebra is necessary. Let $\widetilde{q}, \widetilde{q}\,' \in \mathbb{H}$, if there exists $w \in \mathbb{H} \setminus \{0\}$ such that $\widetilde{q}\,' = w\widetilde{q}w^{-1}$, then we say that $\widetilde{q}$ is equivalent to $\widetilde{q}\,'$, for convenience, we denote it by $\widetilde{q}\,' \sim \widetilde{q}$. Let also $\widetilde{q} = \widetilde{q}_0 + \widetilde{q}_1\,\mathbf{i} + \widetilde{q}_2\,\mathbf{j} + \widetilde{q}_3\,\mathbf{k} \in \mathbb{H}$, we define $\chi : \mathbb{H} \to \mathbb{R}$ by $\chi(\widetilde{q}) = \widetilde{q}_0$ and define the set $[\widetilde{q}] = \{\widetilde{q}\,' \in \mathbb{H} | \widetilde{q}\,' \sim \widetilde{q}\}$, then the following results hold:

(i) $|\widetilde{q}\,'| = |w\widetilde{q}w^{-1}| = |\widetilde{q}|$;

(ii) if $\widetilde{q}\,' \sim \widetilde{q}$, then $\chi(\widetilde{q}\,') = \chi(\widetilde{q})$;

(iii) $[\widetilde{q}] \subset \big\{\widetilde{q}\,' \in \mathbb{H} : \ \chi(\widetilde{q}\,') = \chi(\widetilde{q}), |\widetilde{q}\,'| = |\widetilde{q}|\big\}$;

(iv) if $\widetilde{q} \sim \widetilde{q}_0 + \mathbf{i}\sqrt{\widetilde{q}_1^{\,2} + \widetilde{q}_2^{\,2} + \widetilde{q}_3^{\,2}}$, then

$$[\widetilde{q}] = \Big\{\widetilde{q}_0 + x_1\,\mathbf{i} + x_2\,\mathbf{j} + x_3\,\mathbf{k} : \ x_1^2 + x_2^2 + x_3^2 = \widetilde{q}_1^{\,2} + \widetilde{q}_2^{\,2} + \widetilde{q}_3^{\,2}\Big\}.$$

For any $a \in \mathbb{C}$, we introduce the q-shifted factorial by

$$\begin{cases} (a,q)_n = 1, & n = 0; \\ (a,q)_n = \displaystyle\prod_{l=0}^{n-1}(1 - aq^l), & n \in \mathbb{N}; \\ (a,q)_\infty = \displaystyle\prod_{l=0}^{\infty}(1 - aq^l). \end{cases}$$

Moreover, we obtain

$$(a,q)_\infty = \sum_{n=0}^{\infty}(-1)^n q^{\frac{n(n-1)}{2}}\,\frac{a^n}{(q;q)_n}$$

if $\lim\limits_{n\to\infty}(a;q)_n$ exists.

Now, we will introduce the definition of the q-difference operator.

Definition 2.1 (see [7])**.** Let $f : \overline{q^{\mathbb{N}}} \to \mathbb{H}$, the q-difference operator is defined by

$$D_q f(t) = \begin{cases} \dfrac{f(qt) - f(t)}{qt - t}\,, & t \neq 0, \\ \lim\limits_{n\to\infty} \dfrac{f(q^n) - f(0)}{q^n}\,, & t = 0. \end{cases}$$

The concept of integrable quaternion-valued functions on $\overline{q^{\mathbb{N}}}$ can be introduced naturally as follows.

Definition 2.2 (see [7])**.** Let $f : \overline{q^{\mathbb{N}}} \to \mathbb{H}$, $t, s \in \overline{q^{\mathbb{N}}}$, $f(t) = f^{(0)}(t) + f^{(1)}(t)\,\mathbf{i} + f^{(2)}(t)\,\mathbf{j} + f^{(3)}(t)\,\mathbf{k}$. If $f^{(l)}(t)$ is integrable for $l = 0, 1, 2, 3$, then we say that $f(t)$ is integrable, i.e.,

$$\int_s^t f(t)\,d_q t = \int_s^t f^{(0)}(t)\,d_q t + \int_s^t f^{(1)}(t)\,d_q t\,\mathbf{i} + \int_s^t f^{(2)}(t)\,d_q t\,\mathbf{j} + \int_s^t f^{(3)}(t)\,d_q t\,\mathbf{k}.$$

Now, let $b \in \mathbb{H}$, we define the quaternion q-exponential by

$$e_q^b := \sum_{l=0}^{\infty} \frac{b^l}{(q;q)_l}, \quad l \in \mathbb{N}.$$

Consider the following initial value problem of the quaternion q-difference equation:

$$D_q x(t) = bx(t), \quad x(t_0) = x_0, \tag{2.1}$$

where $b, x_0 \in \mathbb{H}$.

Lemma 2.1 (see [7])**.** *The solution of* (2.1) *with the q-exponential form can be given as*

$$x(t) = e_q^{bt(1-q)} x_0 = \sum_{l=0}^{\infty} \frac{(bt(1-q))^l}{(q;q)_l} x_0.$$

Based on Lemma 2.1, one can obtain the following functions immediately.

Definition 2.3 (see [7])**.** Let $b \in \mathbb{H}$, $t \in \overline{q^{\mathbb{N}}}$, we define the sine and cosine functions by

$$\begin{cases} \sin_q(bt) := \dfrac{1}{2\,\mathbf{i}} \left(e_q^{\mathbf{i}bt(1-q)} - e_q^{-\mathbf{i}bt(1-q)} \right) = \dfrac{1}{\mathbf{i}} \displaystyle\sum_{l=0}^{\infty} \frac{(\mathbf{i}bt(1-q))^{2l+1}}{(q;q)_{2l+1}}, \\ \cos_q(bt) := \dfrac{1}{2} \left(e_q^{\mathbf{i}bt(1-q)} + e_q^{-\mathbf{i}bt(1-q)} \right) = \displaystyle\sum_{l=0}^{\infty} \frac{(\mathbf{i}bt(1-q))^{2l}}{(q;q)_{2l}}. \end{cases}$$

3 Existence and uniqueness of the solution for the higher-order linear quaternion q-difference equations

Consider the higher-order linear quaternion q-difference equations as follows

$$\begin{cases} a_m(t) D_q^m x(t) + a_{m-1}(t) D_q^{m-1} x(t) + \cdots + a_1(t) D_q x(t) + a_0(t) x(t) = B(t), \quad t \in \overline{q^{\mathbb{N}}}, \\ D_q^l x(t_0) = v_l, \end{cases} \tag{3.1}$$

where $a_m, \ldots, a_1, a_0, B : \overline{q^{\mathbb{N}}} \to \mathbb{H}$, $v_l \in \mathbb{H}$, $0 \le l \le m-1$. Let $b_l(t) = a_m^{-1}(t) a_l(t)$, then (3.1) is equivalent to the following q-difference equation

$$\begin{cases} D_q^m x(t) + b_{m-1}(t) D_q^{m-1} x(t) + \cdots + b_1(t) D_q x(t) + b_0(t) x(t) = \widetilde{B}(t), \quad t \in \overline{q^{\mathbb{N}}}, \\ D_q^l x(t_0) = v_l, \end{cases} \tag{3.2}$$

where

$$\widetilde{B}(t) = a_m^{-1}(t) B(t).$$

Below, though applying the transforms $x_0(t) = x(t)$, $x_1(t) = D_q x(t), \ldots, x_{m-1}(t) = D_q^{m-1} x(t)$, one has that (3.2) is equivalent to

$$D_q x_l(t) = f_l\big(t, x_0(t), x_1(t), \ldots, x_{m-1}(t)\big) = \begin{cases} x_{l+1}(t), & l = 0, 1, \ldots, m-2, \\ f(t, x_0(t), x_1(t), \ldots, x_{m-1}(t)), & l = m-1, \end{cases} \tag{3.3}$$

where

$$f\big(t, x_0(t), x_1(t), \ldots, x_{m-1}(t)\big) = \sum_{l=0}^{m-1} \big(-b_l(t) D_q^l x(t) \big) + \widetilde{B}(t), \quad x_l(t_0) = v_l, \ \ 0 \le l \le m-1.$$

Definition 3.1 (see [7])**.** Let $t \in \overline{q^{\mathbb{N}}}$, we define the set $D_l(\mathbb{H})$ as

$$D_l(\mathbb{H}) := \big\{x_l(\,\cdot\,) \in \mathbb{H} :\ |x_l(t) - v_l| < \alpha\big\},$$

where $\{x_l(t)\}_{l=0}^{m-1}$ is continuous at $t = t_0$ and bounded at $t \neq t_0$, i.e., $\lim\limits_{t\to t_0} x_l(t) = x_l(t_0) = v_l \in \mathbb{H}$ and there exists $M > 0$ such that $|x(t)| \le M$ for all $t \in \overline{q^{\mathbb{N}}}$, $\alpha > 0$, $l = 0, 1, \dots, m-1$, $|p| = \sqrt{p_0^2 + p_1^2 + p_2^2 + p_3^2}$ for $p = p_0 + p_1\,\mathbf{i} + p_2\,\mathbf{j} + p_3\,\mathbf{k} \in \mathbb{H}$.

Next, we will establish the solutions of (3.3) with the initial value $x_l(t_0) = v_l$.

Theorem 3.1 (see [7])**.** *Let $t \in \overline{q^{\mathbb{N}}}$, $f_l : \overline{q^{\mathbb{N}}} \times D_0(\mathbb{H}) \times D_1(\mathbb{H}) \times \cdots \times D_{m-1}(\mathbb{H}) \to \mathbb{H}$, $0 \le l \le m-1$. If $f_l(t, x_0(t), x_1(t), \dots, x_{m-1}(t))$ satisfies the following conditions:*

(i) *$f_l(t, x_0(t), x_1(t), \dots, x_{m-1}(t))$ is continuous at $t = t_0$ and boundedat $t \neq t_0$.*

(ii) *$f_l(t, x_0(t), x_1(t), \dots, x_{m-1}(t))$ satisfies the Lipschitz condition, i.e., there exists a constant $K > 0$ such that*

$$\Big|f_l\big(t, x_0(t), x_1(t), \dots, x_{m-1}(t)\big) - f_l\big(t, \widetilde{x}_0(t), \widetilde{x}_1(t), \dots, \widetilde{x}_{m-1}(t)\big)\Big| \le K \sum_{l=0}^{m-1} \big(|x_l(t) - \widetilde{x}_l(t)|\big),$$

where $x_l(\,\cdot\,), \widetilde{x}_l(\,\cdot\,) \in D_l(\mathbb{H})$.

Then (3.3) with the initial value $x_l(t_0) = v_l$ has an unique solution on $[t_0 - h, t_0 + h] \cap \overline{q^{\mathbb{N}}}$, where $h = \min\{\frac{1}{Km(1-q)}, \frac{\alpha}{B}\}$,

$$B := \max_{0 \le l \le m-1} \sup_{|x_l(t) - v_l| < \alpha} \Big|f_l\big(t, x_0(t), x_1(t), \dots, x_{m-1}(t)\big)\Big|, \ \ \alpha > 0.$$

Theorem 3.2 (see [7])**.** *Let $t_0 \in \overline{q^{\mathbb{N}}}$, $h > 1 - q$, where $h = \min\{\frac{1}{Km(1-q)}, \frac{\alpha}{B}\}$. If (3.3) has an initial value $x_l(t_0) = v_l$, then (3.3) has an unique solution on $\overline{q^{\mathbb{N}}}$.*

4 Solving higher-order linear quaternion q-difference equations

In this section, we shall consider the following initial value problem:

$$D_q W(t) = A(t) W(t), \quad W(0) = W_0, \tag{4.1}$$

where $W : \overline{q^{\mathbb{N}}} \to \mathbb{H}^d$, $W_0 \in \mathbb{H}^d$, $d \in \mathbb{N}$, $\mathbb{H}^d$ is the d-dimensional quaternion space, $A(t) = (a_{uv}(t))_{u,v=1}^d \in \mathbb{H}^d$, $a_{uv}(t)$ is continuous at $t = 0$ and bounded at $t \neq 0$.

Theorem 4.1 (see [7])**.** *If $I - (1-q) q^l t A(q^l t)$ is invertible for $l \in \mathbb{N}$. Then the solution of (4.1) can be represented by*

$$W(t) = \prod_{l=0}^{\infty} \big[I - (1-q) q^l t A(q^l t)\big]^{-1} W_0,$$

where $t \in \overline{q^{\mathbb{N}}}$, $A(t)$ is a $d \times d$ quaternion matrix and I is an identity matrix.

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