Some Properties of Topological Entropy of Families of Dynamical Systems on the Cantor Set

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Following [1], we give the definition of topological entropy that will be necessary hereafter. Let X be a compact metric space with a metric d and $f : X \to X$ a continuous mapping. Along with the original metric d, we define an additional system of metrics on X:

$$d_n^f(x,y) = \max_{0\le i\le n-1} d\big(f^i(x), f^i(y)\big), \quad x,y\in X, \quad n\in\mathbb{N},$$

where f^i, $i \in \mathbb{N}$, is the i-th iteration of f, $f^0 \equiv \mathrm{id}_X$. For any $n\in\mathbb{N}$ and $\varepsilon > 0$, denote by $N_d(f,\varepsilon,n)$ the maximum number of points in X, pairwise d_n^f-distances between which are greater than ε. Then *the topological entropy* of the mapping f is defined by the formula

$$h_d(f,x) = \lim_{\varepsilon\to 0}\varlimsup_{n\to\infty}\frac{1}{n}\ln N_d(f,\varepsilon,n).$$

Let $C(X,X)$ denote the set of continuous mappings from X to X with the metric

$$\rho(f,g) = \max_{x\in X} d(f(x), g(x)).$$

Consider the function

$$f \longmapsto h_{\mathrm{top}}(f). \tag{1}$$

It was proved in [2] that function (1) belongs to the second Baire class on the space $C(X,X)$, and the set of points in the space $C(X,X)$ at which function (1) is lower semicontinuous contains an everywhere dense G_δ set. It was established in [3] that the set of points of lower semicontinuity itself is an everywhere dense G_δ set in $C(X,X)$.

If X coincides with the Cantor set $\mathcal{K}$ on the interval $[0,1]$ with the metric induced by the natural metric of the real line, then function (1) is everywhere discontinuous and is lower semicontinuous only at the points where the topological entropy is equal to zero [3]. It was demonstrated in [4] that function (1) does not belong to the first Baire class even on the subspace of homeomorphisms satisfying the Lipschitz condition.

Let us denote by $E_h(f)$ the set of *limiting realizable* values of topological entropy, i.e. those that are obtained for arbitrarily small uniform perturbations of the mapping f:

$$E_h(f) = \bigcap_{n\in\mathbb{N}}\big\{h_{\mathrm{top}}(g):\ \rho(f,g) < n^{-1}\big\}.$$

Theorem 1 ([5])**.** *For each continuous mapping $f : \mathcal{K}\to\mathcal{K}$, the equality $E_h(f) = [0;+\infty]$ holds.*

Given a metric space $\mathcal{M}$ and a continuous mapping $f : \mathcal{M} \to C(X, X)$ let us construct a function

$$\mu \longmapsto h_{\rm top}(f(\mu, \cdot\,). \tag{2}$$

From [2] and [3] it follows that the set of points in the space $\mathcal{M}$ at which function (2) is lower semicontinuous is an everywhere dense G_δ set. In the case $\mathcal{M} = X = \mathcal{K}$ for any everywhere dense G_δ set $A \subset \mathcal{M}$, there is a continuous mapping $f : \mathcal{M} \to C(X, X)$ such that $h_{\rm top}(f(A, \cdot\,) = 0$ and $h_{\rm top}(f(\mathcal{M} \setminus A, \cdot\,) = +\infty$ [5]. In particular, the set of points of lower semicontinuity of function (2) coincides with the set A. It turns out that using the method of [5] one can prove the following

Theorem 2. *If $\mathcal{M} = X = \mathcal{K}$, then for any number $h > 0$ and an everywhere dense G_δ set $A \subset \mathcal{M}$, there is a continuous mapping $f : \mathcal{M} \to C(X, X)$ such that the equalities $h_{\rm top}(f(A, \cdot\,)) = 0$ and $h_{\rm top}(f(\mathcal{M} \setminus A, \cdot\,)) = h$ are satisfied.*

References

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