Existence and Asymptotic Behavior of Nonoscillatory Solutions of Quasilinear Differential Equations with Variable Exponents

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1 Introduction

Recently, there has been an increasing interest in studying differential equation with variable exponents, that is, differential equations with $\alpha(t)$-Laplacian (Generally referred to as $p(t)$-Laplacian and see below for the details) of the form

$$\big(p(t)\varphi_{\alpha(t)}(x')\big)' + q(t)\varphi_{\beta(t)}(x) = 0, \ \ t \geqq a, \tag{A}$$

where $\alpha(t)$, $\beta(t)$, $p(t)$ and $q(t)$ are positive continuous functions on $[a, \infty)$, $a \geqq 1$ and use is made of the notation

$$\varphi_{\gamma(t)}(\xi) = |\xi|^{\gamma(t)-1}\xi = |\xi|^{\gamma(t)} \operatorname{sgn} \xi, \ \ \xi \in \mathbb{R}, \ \ \gamma \in C[1, \infty).$$

By a solution of (A) we mean a function $x \in C^1[T_x, \infty)$, $T_x \geqq a$, which has the property $p(t)\varphi_{\alpha(t)}(x') \in C^1[T_x, \infty)$ and satisfies the equation at all points $t \geqq T_x$. A nontrivial solution $x(t)$ of (A) is said to be nonoscillatory if $x(t) \neq 0$ for all large t and oscillatory otherwise. In this talk we restrict our attention to its eventually positive solutions.

The first interest in $\alpha(\,\cdot\,)$-type Laplacian (i.e., $p(\,\cdot\,)$-type Laplacian) was in function spaces called variable exponent spaces. Variable exponent spaces appeared in the literature for the first time in a 1931 paper by Orlicz ([9]). In 2000 and 2011, Růžička and Diening et al. studied equation with non-standard $p(x)$-growth in the modeling of the so-called electrorheological fluids ([10]) and the Lebesgue and Sobolev spaces with variable exponetns([2]), respectively. The mathematically and physically importance of $p(\,\cdot\,)$-type Laplacian was recognized by the above-mentioned Růžička's monograph (see [8]).

In recent years, there has been well analyzed the oscillatory and nonoscillatory behavior of the equation with $p(t)$-Laplacian

$$\big(a(t)\varphi_{p(t)-1}(x')\big)' \pm b(t)\varphi_{q(t)-1}(x) = 0, \ \ t \geqq a, \ \ (p(t) = q(t) \ \text{ or } \ p(t) \neq q(t)),$$

which is of the same type as (A) but written in a different representation of $p(t) = a(t)$, $q(t) = b(t)$, $\alpha(t) = p(t) - 1$ $(p(t) > 1)$ and $\beta(t) = q(t) - 1$ $(q(t) > 1)$ in equation (A) (see [1, 3–7]).

To the best of the author's knowledge, detail is unknown about nonoscillatory behavior of (A), and so in this talk we make an attempt to investigate in detail the existence and asymptotic behavior of eventually positive solutions of (A).

2 Existence of positive solutions

In this talk we make the following assumptions without further mentioning:

$$\int_a^\infty \Big[\frac{k}{p(t)}\Big]^{\frac{1}{\alpha(t)}} dt = \infty \tag{2.1}$$

for every constant $k > 0$, and employ the notation

$$P_{\alpha(t),k}(t) = \int\limits_T^t \Big[\frac{k}{p(s)}\Big]^{\frac{1}{\alpha(s)}}\,ds, \ \ t \geqq T \geqq a. \tag{2.2}$$

From (2.1) and (2.2) it is obvious that

$$P_{\alpha(T),k}(T) = 0, \ \ \lim_{t\to\infty} P_{\alpha(t),k}(t) = \infty \ \text{ for every } \ k > 0,$$
$$P_{\alpha(t),k}(t) > P_{\alpha(t),l}(t), \ \ t > T \ \text{ for } \ k > l > 0 \ \text{ and } \ \lim_{k\to 0} P_{\alpha(t),k}(t) = 0 \ \text{ for each } \ t \geqq T.$$

First of all, we begin by classifying all possible positive solutions of equation (A) according to their asymptotic behavior as $t \to \infty$.

Lemma 2.1. *One and only one of the following cases occurs for each positive solution $x(t)$ of* (A)*:*

I. $\lim\limits_{t\to\infty} p(t)\varphi_{\alpha(t)}(x'(t)) = const. > 0$, $\lim\limits_{t\to\infty} x(t) = \infty$;

II. $\lim\limits_{t\to\infty} p(t)\varphi_{\alpha(t)}(x'(t)) = 0$, $\lim\limits_{t\to\infty} x(t) = \infty$;

III. $\lim\limits_{t\to\infty} p(t)\varphi_{\alpha(t)}(x'(t)) = 0$, $\lim\limits_{t\to\infty} x(t) = const > 0$.

We want to obtain criteria for the existence of positive solutions of (A) of type I, II and III.

Theorem 2.1. *Suppose that for each fixed $k > 0$ and $T \geqq a$,*

$$\lim_{l\to 0} \frac{P_{\alpha(t),l}(t)}{P_{\alpha(t),k}(t)} = 0 \tag{2.3}$$

uniformly on any interval of the form $[T_1,\infty)$, $T_1 > T$. Then equation (A) *possesses a positive solution of type* I *if and only if*

$$\int\limits_a^\infty q(t)(P_{\alpha(t),k}(t))^{\beta(t)}\,dt < \infty \ \text{ for some constant } \ k > 0. \tag{2.4}$$

Theorem 2.2. *Equation* (A) *possesses a positive solution of type* III *if and only if*

$$\int\limits_a^\infty \bigg[\frac{1}{p(t)}\int\limits_t^\infty q(s)c^{\beta(s)}\,ds\bigg]^{\frac{1}{\alpha(t)}}\,dt < \infty \ \text{ for some constant } \ c > 0. \tag{2.5}$$

Unlike the solution of the types I and III it is difficult to characterize the type II solution of (A), and so we content ourselves with sufficient conditions for the existence of such solutions of (A).

Theorem 2.3. *Suppose that* (2.3) *holds. Equation* (A) *possesses a positive solution of type* II *if* (2.4) *holds for some constant $k > 0$ and*

$$\int\limits_a^\infty \bigg[\frac{1}{p(t)}\int\limits_t^\infty q(s)d^{\beta(s)}\,ds\bigg]^{\frac{1}{\alpha(t)}}\,dt = \infty \tag{2.6}$$

for every constant $d > 0$.

3 Examples

We now present some examples illustrating Theorem 2.1 obtained in Section 2.

Example 3.1. Consider the equations with variable exponents of nonlinearity

$$\big(e^{-(t^2-1)}\varphi_t(x')\big)' + q_1(t)\varphi_t(x) = 0, \quad t \geqq e, \tag{E$_1$}$$

$$\big(e^{-(t^2-1)}\varphi_t(x')\big)' + q_2(t)\varphi_{\frac{1}{t}}(x) = 0, \quad t \geqq e, \tag{E$_2$}$$

and

$$\big(e^{-(1-\frac{1}{t^2})}\varphi_{\frac{1}{t}}(x')\big)' + q_3(t)\varphi_{\frac{1}{t}}(x) = 0, \quad t \geqq e, \tag{E$_3$}$$

where the functions $q_i(t)$, $i = 1, 2, 3$, are

$$q_1(t) = e^{-(t^2-1)}\Big(1+\frac{1}{t^2}\Big)^t\Big\{\frac{2}{t^2+1} - \log\Big(1+\frac{1}{t^2}\Big)\Big\},$$
$$q_2(t) = e^{-(1-\frac{1}{t^2})}\Big(1+\frac{1}{t^2}\Big)^t\Big\{\frac{2}{t^2+1} - \log\Big(1+\frac{1}{t^2}\Big)\Big\},$$

and

$$q_3(t) = e^{-(1-\frac{1}{t^2})}\frac{1}{t^2}\Big(1+\frac{1}{t^2}\Big)^{\frac{1}{t}}\Big\{\frac{2}{t^2+1} + \log\Big(1+\frac{1}{t^2}\Big)\Big\},$$

respectively. They are special cases of (A) with $\alpha(t) = t$ in (E$_i$), $i = 1, 2$, $\alpha(t) = 1/t$ in (E$_3$), $\beta(t) = t$ in (E$_1$), $\beta(t) = 1/t$ in (E$_i$), $i = 2, 3$, $p(t) = e^{-(t^2-1)}$ in (E$_i$), $i = 1, 2$, $p(t) = e^{-(1-\frac{1}{t^2})}$ in (E$_3$) and $q(t) = q_i(t)$, $i = 1, 2, 3$ in the above. The functions $p(t) = e^{-(t^2-1)}$ and $p(t) = e^{-(1-\frac{1}{t^2})}$ satisfy (2.1) with $k = 1$ and, in addition, the function $P_{\alpha(t),1}(t)$ associated with (E$_i$), $i = 1, 2, 3$ is

$$P_{\alpha(t),1}(t) = \int_e^t \Big[\frac{1}{p(s)}\Big]^{\frac{1}{\alpha(s)}} ds = \int_e^t e^{s-\frac{1}{s}}\, ds \ \sim\ e^{t-\frac{1}{t}} \ \text{ as } \ t\to\infty$$

by (2.2), where the symbol $\sim$ is used to denote the asymptotic equivalence

$$f(t) \sim g(t) \ \text{ as } \ t\to\infty \iff \lim_{t\to\infty}\frac{f(t)}{g(t)} = 1.$$

Since

$$\int_e^\infty q_1(t)(P_{t,1}(t))^t\, dt = \int_e^\infty e^{-(t^2-1)}\Big(1+\frac{1}{t^2}\Big)^t\Big\{\frac{2}{t^2+1} - \log\Big(1+\frac{1}{t^2}\Big)\Big\}(e^{t-\frac{1}{t}})^t\, dt$$
$$= \int_e^\infty \Big(1+\frac{1}{t^2}\Big)^t\Big\{\frac{2}{t^2+1} - \log\Big(1+\frac{1}{t^2}\Big)\Big\}\, dt < \infty,$$
$$\int_e^\infty q_2(t)(P_{t,1}(t))^{\frac{1}{t}}\, dt = \int_e^\infty \Big(1+\frac{1}{t^2}\Big)^t\Big\{\frac{2}{t^2+1} - \log\Big(1+\frac{1}{t^2}\Big)\Big\}\, dt < \infty,$$

and

$$\int_e^\infty q_3(t)(P_{\frac{1}{t},1}(t))^{\frac{1}{t}}\, dt = \int_e^\infty \frac{1}{t^2}\Big(1+\frac{1}{t^2}\Big)^{\frac{1}{t}}\Big\{\frac{2}{t^2+1} + \log\Big(1+\frac{1}{t^2}\Big)\Big\}\, dt < \infty,$$

we can apply Theorem 2.1 to conclude that there exists a positive solution of type I such that $x(t) = e^{t-\frac{1}{t}}$, which satisfies

$$\lim_{t\to\infty} p(t)\varphi_t(x'(t)) = \lim_{t\to\infty} \Big(1+\frac{1}{t^2}\Big)^t = 1, \quad \lim_{t\to\infty} x(t) = \infty$$

for (E_i), $i = 1, 2$ and that for (E_3)

$$\lim_{t\to\infty} p(t)\varphi_{\frac{1}{t}}(x'(t)) = \lim_{t\to\infty} \Big(1+\frac{1}{t^2}\Big)^{\frac{1}{t}} = 1, \quad \lim_{t\to\infty} x(t) = \infty.$$

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