Periodic Solutions in Dynamic Equations on Time Scales and Their Relationship with Differential Equations

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1 Basic concepts of the theory of time scales

A time scale, denoted by $\mathbb{T}$, is defined as an arbitrary nonempty closed subset of the real axis. To refer to a subset of the time scale, we use the notation $A_{\mathbb{T}}$, where $A_{\mathbb{T}}$ represents the intersection of set A of the real axis with the time scale $\mathbb{T}$.

For every time scale there are defined two operators, the forward jump operator σ and the backward jump operator ρ, which are integral to this theory. The forward jump operator is defined as $\sigma(t)=\inf\{s\in\mathbb{T}:\ \ s>t\}$, while the backward jump operator is defined as $\rho(t)=\sup\{s\in\mathbb{T}:\ \ s<t\}$. It's important to note that in this context, we assume that $\inf\varnothing:=\sup\mathbb{T}$ and $\sup\varnothing:=\inf\mathbb{T}$.

A key component of time scale theory is the graininess function, denoted as μ, which maps elements of the time scale to the interval $[0,\infty]$. It is defined as $\mu(t)=\sigma(t)-t$.

Additionally, a point $t\in\mathbb{T}$ is characterized as left-dense (LD), left-scattered (LS), right-dense (RD), or right-scattered (RS) based on conditions involving the operators ρ and σ. If $\mathbb{T}$ contains a left-scattered maximum M, we define $\mathbb{T}^k=\mathbb{T}\setminus M$; otherwise, $\mathbb{T}^k=\mathbb{T}$.

Moreover, a function $f:\mathbb{T}\to\mathbb{R}^n$ is considered Δ-differentiable at $t\in\mathbb{T}^k$ if the limit

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(\sigma(t))-f(t)}{\sigma(t)-t}$$

exists in $\mathbb{R}^n$.

Let us recall the following classical results (see [1,2]):

(a) If $t\in\mathbb{T}^k$ is a right-dense point of $\mathbb{T}$, then f is Δ-differentiable at t if and only if the limit

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}$$

exists in $\mathbb{R}^n$.

(b) If $t\in\mathbb{T}^k$ is a right-scattered point of $\mathbb{T}$, and if f is continuous at t, then f is Δ-differentiable at t, and

$$f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}\,.$$

To delve into the properties of periodicity in dynamic equations on time scales, it is essential to establish a clear understanding of periodicity on these time scales.

We say that a time scale $\mathbb{T}$ is called a periodic time scale if

$$\Pi := \big\{\tau \in \mathbb{R} : \ \ t \pm \tau \in \mathbb{T}, \ \ \forall\, t \in \mathbb{T}\big\} \neq \{0\}.$$

The smallest positive $\tau \in \Pi$ is called the period of the time scale.

Definition 1.1 ([5])**.** Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period τ. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic if there exists a natural number n such that $P = n\tau$, $f(t+P) = f(t)$ for all $t \in \mathbb{T}$. The smallest positive number P is called the period of function f if $f(t+P) = f(t)$ for all $t \in \mathbb{T}$.

If $\mathbb{T} = \mathbb{R}$, we say that f is periodic with period $P > 0$, if P is the smallest positive number such that $f(t+P) = f(t)$ for all $t \in \mathbb{T}$.

It's worth noting that if $\mathbb{T}$ is a periodic time scale with period τ, then the forward jump operator σ exhibits periodic behavior, where $\sigma(t+n\tau) = \sigma(t)+n\tau$. This periodicity extends to the graininess function, as

$$\mu(t+n\tau) = \sigma(t+n\tau) - (t+n\tau) = \sigma(t) - t = \mu(t).$$

In our subsequent study, we consider a set of periodic time scales denoted as $\mathbb{T}_\lambda$, where $\lambda \in \Lambda \subset \mathbb{R}$ and $\lambda = 0$ serves as a limit point of the set Λ. It is assumed that for any $\lambda \in \Lambda$, $\inf \mathbb{T}_\lambda = -\infty$, $\sup \mathbb{T}_\lambda = \infty$, and the point $t = 0$ is a part of $\mathbb{T}_\lambda$ for all $\lambda \in \Lambda$.

Let $\mathbb{T}_\lambda$ be a periodic time scale with period $\tau_\lambda = \frac{\omega}{n(\lambda)}$, where $n(\lambda)$ is a natural number. We set $\mu_\lambda := \sup\limits_{t \in \mathbb{T}_\lambda} \mu_\lambda(t)$, where $\mu_\lambda(t) : \mathbb{T}_\lambda \to [0,\infty)$ is the graininess function. If $\mu_\lambda \to 0$ as $\lambda \to 0$, then $\mathbb{T}_\lambda$ converges to the continuous time scale $\mathbb{T}_0 = \mathbb{R}$, and the dynamic equation system on the time scale transforms into the corresponding system of differential equations. Due to the periodicity of the graininess function $\mu\lambda(t)$, on each subset of the time scale $[t; t+\tau]_\lambda \subset \mathbb{T}_\lambda$, the following equality holds:

$$\sup_{t \in [t; t+\tau]_\lambda} \mu_\lambda(t) = \mu_\lambda.$$

Hence, it is naturaly to expect that, under certain conditions, the existence of a periodic solution in a differential equation implies the existence of a corresponding solution in the dynamic equation on the periodic time scale $\mathbb{T}_\lambda$, and vice versa.

2 Problem statement and auxiliary results

We consider the system of differential equations

$$\frac{dx}{dt} = X(t,x), \tag{2.1}$$

where $x \in D$, $D \subset \mathbb{R}^n$ is a domain in the space $\mathbb{R}^d$, and the corresponding system of equations defined on $\mathbb{T}_\lambda$

$$x_\lambda^\Delta = X(t, x_\lambda), \tag{2.2}$$

where $t \in \mathbb{T}_\lambda$, $\lambda \in \Lambda \subset \mathbb{R}$, $\lambda = 0$ is a limit point of the set Λ, $x_\lambda : \mathbb{T}_\lambda \to \mathbb{R}^n$, and $x_\lambda^\Delta(t)$ is the Δ-derivative of $x_\lambda(t)$ in $\mathbb{T}_\lambda$.

Assume that $X(t,x)$ is continuously differentiable and bounded with its partial derivatives, i.e. there exists $C > 0$ such that

$$|X(t,x)| + \Big|\frac{\partial X(t,x)}{\partial t}\Big| + \Big\|\frac{\partial X(t,x)}{\partial x}\Big\| \le C$$

for $t \in \mathbb{R}$ and $x \in D$. Here $\frac{\partial X}{\partial x}$ is the corresponding Jacobian matrix, $|\cdot|$ is the Euclidian norm of a vector, and $\|\cdot\|$ is the norm of a matrix.

In addition, we also assume that the function $X(t,x)$ is periodic in t with a period ω, i.e.

$$X(t+\omega, x) = X(t,x), \ \ t \in \mathbb{R}, \ \ x \in D.$$

We need a lemma to address the evaluation of the discrepancy between the solutions of a Cauchy problem for a system of differential equations and the corresponding solutions of dynamic equations on time scales, given that they share the same initial conditions.

Lemma 2.1 ([4])**.** *Let $t_0 \in \mathbb{T}_\lambda$, $t_0+T \in \mathbb{T}_\lambda$, x_λ and $x(t)$ are the solutions of* (2.2) *and* (2.1) *on* $[t_0, t_0+T]$ *and* $[t_0,t_0+T]_{\mathbb{T}_\lambda}$*, respectively. Then if the initial conditions $x(t_0) = x_\lambda(t_0) = x_0$, $x_0 \in D$ are satisfied, the following inequality holds*

$$|x(t) - x_\lambda(t)| \le \mu(\lambda) K(T),$$

where

$$\mu(\lambda) = \sup_{t \in [t_0, t_0+T]_{\mathbb{T}_\lambda}} \mu_\lambda(t) \ \ \textit{for} \ \ t \in [t_0, t_0+T]_{\mathbb{T}_\lambda},$$
$$K(T) = e^{C(T+1)}\Big(C + \frac{C^2 T}{4}\Big) + 3C \ \ \textit{is constant.}$$

Let us define the notion of asymptotic stability for solutions of dynamic equations on time scales, drawing parallels with the definition of asymptotic stability in the context of differential equations as outlined in [3].

Definition 2.1. A solution $x_\lambda(t)$ of system (2.2), defined on a family of time scales $\mathbb{T}_\lambda$, is called uniformly in t_0 and λ asymptotically stable if for any $\varepsilon > 0$ there exist $\delta > 0$ and $T > 0$, which do not depend on t_0 and λ, such that if $y_\lambda(t)$ is a solution of system (2.2) and

$$|x_\lambda(t_0) - y_\lambda(t_0)| < \delta,$$

then

$$|x_\lambda(t) - y_\lambda(t)| < \varepsilon, \ \ \text{if} \ \ t \ge t_0,$$
$$|x_\lambda(t) - y_\lambda(t)| \le \frac{\delta}{2}, \ \ \text{if} \ \ t \in [t_0+T, \infty)_{\mathbb{T}_\lambda}.$$

3 Main results

We were considering the sequence of periodic time scales $\mathbb{T}_\lambda$ with the smallest period $\tau(\lambda)$ such that $\tau(\lambda)$ approaches 0 as $\lambda \to 0$, and $\frac{\tau(\lambda)}{\omega}$ is rational. And we delineated prerequisites for the existence of a periodic solution to the system described by equation (2.1), provided that the system (2.2) already possesses a periodic solution.

Theorem 3.1. *Suppose there exists a positive value λ_0 such that for all λ less than λ_0, the system of differential equations* (2.2) *has a uniformly along $t_0 \in \mathbb{T}_\lambda$ and λ asymptotically stable periodic solution $x_\lambda(t)$, which belongs to the domain D along with ρ-neighborhood. Then the dynamic system* (2.1) *also has a periodic solution with period $p = r\omega$, where r is an integer.*

Proof. Since $x_\lambda(t)$ is asymptotically stable, then for any $\varepsilon > 0$ ($\varepsilon < \frac{\rho}{2}$) there exist $\delta > 0$ ($\delta < \varepsilon$) and $\widetilde{T} > 0$, which are independent of t_0 and λ, such that if

$$|x_\lambda(t_0) - y_\lambda(t_0)| \le \delta,$$

then

$$|x_\lambda(t) - y_\lambda(t)| < \varepsilon, \ \text{ if } \ t \ge 0, \tag{3.1}$$

$$|x_\lambda(t) - y_\lambda(t)| \le \frac{\delta}{2}\,, \ \text{ if } \ t \in [\widetilde{T}, \infty)_\lambda. \tag{3.2}$$

Without loss of generality, it can be assumed that $t_0(\lambda) = 0$. Let T be the smallest point right from $\widetilde{T}$ such that $T = r_0\omega$, r_0 is an integer.

Let us choose λ_0 such that for any $\lambda < \lambda_0$ and for the defined $\delta > 0$ and T the following conditions hold:

(1) the corresponding time scale $\mathbb{T}_\lambda$ with the graininess function μ_λ has the period $\tau_\lambda = \frac{\omega}{m_0}$, m_0 is an integer;

(2) if $y_\lambda(t)$ is a solution of the dynamic system (2.2) on time scale $\mathbb{T}_\lambda$ and $\varphi(t)$ is a solution of the differential system (2.1) such that $\varphi(t_k) = y_\lambda(t_k)$, $t_k \in \mathbb{T}_\lambda$, then the following inequality holds:
$$|\varphi(t) - y_\lambda(t)| < \frac{\delta}{2}\,, \ \ t \in [t_k, t_{k+1}]_\lambda, \tag{3.3}$$
where t_{k+1} is the smallest point in the interval $[t_k + T, t_k + T + 1]_{\mathbb{T}_\lambda}$ such that $t_{k+1} = i_{k+1}\tau_\lambda$, with $i_{k+1} \in \mathbb{N}$. As $\lambda \to 0$ both μ_λ and τ_λ tend to zero, which ensures the existence of such a point for sufficiently small graininess function.

Since we can choose λ_0 such that for any $\lambda < \lambda_0$ it holds $\mu_\lambda K(T+1) \le \delta/2$, then, by Lemma 2.1, the inequality (3.3) holds.

For the corresponding μ_λ, according to the conditions of Theorem 3.1 and Definition 1.1, the system (2.2) has a periodic asymptotically stable solution $x_\lambda(t)$ with a period $P_\lambda = n_0\tau_\lambda$.

We consider the δ-neighborhood of the point $x_\lambda(0)$. Let y_0 be any point in this neighborhood. Then

$$|x_\lambda(0) - y_0| \le \delta.$$

Let $\varphi(t, y_0)$ be a solution of the system (2.1), and let $y_\lambda(t)$ be a solution of the system (2.2), both satisfying the initial condition $\varphi(0, y_0) = y_\lambda(0) = y_0$ at the point $t_0(\lambda) = 0$.

Let's consider the interval $[0, T]_\lambda$. Since $T = r_0\omega$, $\omega = m_0\tau_\lambda$ and $i_1 := r_0 m_0$, then $T = r_0 m_0 \tau_\lambda = 0 + i_1\tau_\lambda = t_1 \in \mathbb{T}_\lambda$. So, from the inequalities (3.1) and (3.2) it follows that

$$|y_\lambda(T) - x_\lambda(T)| \le \frac{\delta}{2}\,.$$

Consequently, considering (3.2), (3.3), we obtain

$$|x_\lambda(T) - \varphi(T, y_0)| \le |x_\lambda(T) - y_\lambda(T)| + |y_\lambda(T) - \varphi(T, y_0)| < \delta.$$

Thus, the solution $\varphi(t)$ of the system (2.1), which starts in the δ-neighborhood of $x_\lambda(0)$, does not leave the 2ε-neighborhood of the solution $x_\lambda(t)$ of the system (2.2) on the interval $[0, T]_\lambda$ of time scale $\mathbb{T}_\lambda$, returns to the δ-neighborhood of $x_\lambda(t)$ at time T, provided that the solution $\varphi(t)$ is defined on the interval $[0, T]$.

Let $\widehat{y}_\lambda(t)$ be a solution of the system (2.2) such that its initial conditions coincide with the initial conditions of the solution $\varphi(t)$ at time T:

$$\varphi(T) = \widehat{y}_\lambda(T).$$

Let's consider the interval $[T, 2T]_{\mathbb{T}_\lambda}$. If $i_2 := 2r_0m_0$, then we get

$$2T = 2r_0m_0\tau_\lambda = 0 + i_2\tau_\lambda = t_2 \in \mathbb{T}_\lambda.$$

So,

$$|\widehat{y}_\lambda(2T) - x_\lambda(2T)| \leq \frac{\delta}{2}$$

and we have

$$|x_\lambda(2T) - \varphi(2T)| < \delta.$$

Continuing this process, we obtain on each interval $[(k-1)T, kT]$

$$|x_\lambda(kT) - \varphi(kT)| < \delta.$$

Recall that the time scale T_λ has a period $\tau_\lambda = \frac{\omega}{m_0}$, and, according to the definition 1.1, the solution x_λ has a period $P_\lambda = n_0\tau_\lambda$.

Then, at the point $t_{k_M} = M\tau_\lambda := r\omega$ from the set of points $\{t_k = kT\}$, r is divisible by n_0, we have:

$$|x_\lambda(t_{k_M}) - \varphi(r\omega)| < \delta,$$

where M is a common multiple of m_0, r_0 and n_0.

Because $x_\lambda(t_{k_M}) = x_\lambda(0)$, $\pi : y_0 \to \varphi(r\omega, y_0)$ maps the ball of radius δ onto itself. Thus, there exists a fixed point y_1 of the mapping π such that

$$\varphi(r\omega, y_1) = y_1.$$

This implies that the solution of the system (2.1) with the initial condition $\varphi(0) = y_1$ is periodic with a period $r\omega$, which completes the proof. □

The next theorem establishes the existence of a periodic solution of the system (2.2) on $\mathbb{T}_\lambda$, if the system (2.1) has the corresponding periodic solution.

Theorem 3.2. *Suppose the system of dynamic equations* (2.1) *has an asymptotically stable periodic solution $x(t)$ with a period ω, which belongs to the domain D with ρ-neighborhood. Then there exist $\lambda_0 > 0$ such that for all $\lambda < \lambda_0$ the differential system* (2.2) *has at least one periodic solution with a period $r\omega$ on $\mathbb{T}_\lambda$, where r is an integer.*

Acknowledgement

The work of Oleksandr Stanzhytskyi was partially supported by the National Research Foundation of Ukraine # F81/41743 and by the Ukrainian Government Scientific Research Grant # 210BF38-01.

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