Solvability of BVPs for Sequential Fractional Differential Equations at Resonance

Svatoslav Staněk

Department of Mathematical Analysis, Faculty of Science, Palacký University Olomouc, Czech Republic

E-mail: svatoslav.stanek@upol.cz

1 Introduction

Let $T > 0$ be given, $J = [0, T]$ and $\|x\| = \max\{|x(t)| : t \in J\}$ be the norm in $C(J)$.

We discuss the fractional boundary value problem

$$^{c}\!D^{\alpha}x(t) - p(t, x(t))\,^{c}\!D^{\alpha-1}x(t) = f(t, x(t)), \tag{1.1}$$

$$x(0) = x(T), \quad x'(0) = 0, \tag{1.2}$$

where $\alpha \in (1, 2]$, $p, f \in C(J \times \mathbb{R})$ and $^{c}\!D$ denotes the Caputo fractional derivative.

Definition 1.1. We say that $x : J \to \mathbb{R}$ is *a solution of equation* (1.1) if $x', {}^{c}\!D^{\alpha}x \in C(J)$ and (1.1) holds for $t \in J$. A solution x of (1.1) satisfying the boundary condition (1.2) is called *a solution of problem* (1.1), (1.2).

The special case of (1.1) is the differential equation $x'' - p(t, x)x' = f(t, x)$. Problem (1.1), (1.2) is at resonance, because each constant function x on J is a solution of problem $^{c}\!D^{\alpha}x - p(t, x)\,^{c}\!D^{\alpha-1}x = 0$, (1.2).

The aim of this paper is to give conditions guaranteeing the existence and uniqueness of solutions to problem (1.1), (1.2). It is shown that this problem is reduced to the existence of a fixed point of an integral operator $\mathcal{S}$ in the set $C(J) \times \mathbb{R}$. The Schaefer fixed point theorem [1] is applied for solving $\mathcal{S}(x, c) = (x, c)$.

2 Preliminaries

We recall the definitions of the Riemann–Liouville fractional integral and the Caputo fractional derivative [2, 3].

The Riemann–Liouville fractional integral $I^{\gamma}x$ of order $\gamma > 0$ of a function $x : J \to \mathbb{R}$ is defined as

$$I^{\gamma}x(t) = \int_0^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\, x(s)\, \mathrm{d}s,$$

where Γ is the Euler gamma function. I^0 is the identical operator.

The Caputo fractional derivative $^{c}\!D^{\gamma}x$ of order $\gamma > 0$, $\gamma \notin \mathbb{N}$, of a function $x : J \to \mathbb{R}$ is given as

$$^{c}\!D^{\gamma}x(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{(t-s)^{n-\gamma-1}}{\Gamma(n-\gamma)} \Big(x(s) - \sum_{k=0}^{n-1} \frac{x^{(k)}(0)}{k!} s^k\Big)\, \mathrm{d}s,$$

where $n = [\gamma]+1$, $[\gamma]$ means the integral part of the fractional number γ. If $\gamma \in \mathbb{N}$, then ${}^cD^\gamma x = x^{(\gamma)}$. In particular,

$$ {}^cD^\gamma x(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}(x(s)-x(0))\,\mathrm{d}s = \frac{\mathrm{d}}{\mathrm{d}t} I^{1-\gamma}(x(t)-x(0)) \ \text{ if } \ \gamma \in (0,1). $$

Let $\mathcal{P}, \mathcal{F} : C(J) \to C(J)$ be the Nemytskii operators associated to p, f,

$$ \mathcal{P}x(t) = p(t, x(t)), \quad \mathcal{F}x(t) = f(t, x(t)). $$

Equation (1.1) can be written as

$$ {}^cD^\alpha x(t) - \mathcal{P}x(t) {}^cD^{\alpha-1}x(t) = \mathcal{F}x(t). $$

Let an operator $\mathcal{Q}$ acting on $C(J)$ be defined by the formula

$$ \mathcal{Q}x(t) = \int_0^t \mathcal{F}x(s)\exp\left(\int_s^t \mathcal{P}x(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s, \quad t \in J. $$

Then $\mathcal{Q}x(t)|_{t=0} = 0$, $\mathcal{Q} : C(J) \to C^1(J)$ and for $x \in C(J)$, $t \in J$,

$$ (\mathcal{Q}x(t))' = \mathcal{P}x(t)\mathcal{Q}x(t) + \mathcal{F}x(t), $$

$$ I^{\alpha-1}\mathcal{Q}x(t) = I^\alpha(\mathcal{Q}x(t))' = I^\alpha(\mathcal{P}x(t)\mathcal{Q}x(t) + \mathcal{F}x(t)). \tag{2.1} $$

The following result deals with solutions x of equation (1.1) satisfying the initial condition

$$ x(0) = c, \quad x'(0) = 0, \tag{2.2} $$

where $c \in \mathbb{R}$.

Lemma 2.1. *If x is a solution of the initial value problem* (1.1), (2.2), *then*

$$ x(t) = c + I^{\alpha-1}\mathcal{Q}x(t), \quad t \in J. \tag{2.3} $$

Also vice versa if $x \in C(J)$ satisfies (2.3), *then x is a solution of problem* (1.1), (2.2).

Let $\mathcal{S} : C(J) \times \mathbb{R} \to C(J) \times \mathbb{R}$ be an operator defined by

$$ \mathcal{S}(x, c) = \left(c + I^{\alpha-1}\mathcal{Q}x(t), c - I^{\alpha-1}\mathcal{Q}x(t)\big|_{t=T}\right). $$

The relation between fixed points of $\mathcal{S}$ and solutions of problem (1.1), (1.2) is given in the following result.

Lemma 2.2. *If $(x, c) \in C(J) \times \mathbb{R}$ is a fixed point of $\mathcal{S}$, then x is a solution of problem* (1.1), (1.2) *and $c = x(0)$. If x is a solution of problem* (1.1), (1.2), *then $(x, x(0)) \in C(J) \times \mathbb{R}$ is a fixed point of $\mathcal{S}$.*

Proof. Let $(x, c) \in C(J) \times \mathbb{R}$ be a fixed point of $\mathcal{S}$. Then

$$ x(t) = c + I^{\alpha-1}\mathcal{Q}x(t), \quad t \in J, \tag{2.4} $$

$$ I^{\alpha-1}\mathcal{Q}x(t)\big|_{t=T} = 0. \tag{2.5} $$

Now we conclude from Lemma 2.1 and equality (2.4) that x is a solution of (1.1) and $x(0) = c$, $x'(0) = 0$. The equality $x(T) = c$ follows from (2.4) and (2.5). Hence x is a solution of problem (1.1), (1.2).

Let x be a solution of problem (1.1), (1.2) and let $x(0) = c$. Then (see (1.2)) $x(T) = c$. By Lemma 2.1, x satisfies equality (2.3) which together with $x(T) = c$ gives $I^{\alpha-1}\mathcal{Q}x(t)|_{t=T} = 0$. Consequently, (x, c) is a fixed point of $\mathcal{S}$. □

Lemma 2.3. *Operator $\mathcal{S}$ is completely continuous.*

3 Existence results

Theorem 3.1. *Let*

(H_1) *$p(t,x)$ be bounded and nonnegative on $J \times \mathbb{R}$,*

(H_2) *there exist $D > 0$ such that*

$$xf(t,x) > 0 \ \text{ for } \ t \in J, \ \ |x| \geq D,$$

(H_3) *there exist $A, B \in [0,\infty)$ such that*

$$|f(t,x)| \leq A + B|x| \ \text{ for } \ t \in J, \ \ x \in \mathbb{R}.$$

Then problem (1.1), (1.2) *has at least one solution. In addition, $|x(0)| < D$ for each solution x of this problem.*

Proof. Keeping in mind Lemma 2.2, we need to prove that the operator $\mathcal{S}$ admits a fixed point in $C(J) \times \mathbb{R}$. Since $\mathcal{S}$ is a completely continuous operator by Lemma 2.3, the Schaefer fixed point theorem guarantees the existence of a fixed point of $\mathcal{S}$ if the set

$$\mathcal{M} = \Big\{(x,c) \in C(J) \times \mathbb{R} : \ \ (x,c) = \lambda\mathcal{S}(x,c) \ \text{ for some } \ \lambda \in (0,1)\Big\}$$

is bounded in $C(J) \times \mathbb{R}$.

In order to prove the boundedness of $\mathcal{M}$, let $(x,c) = \lambda\mathcal{S}(x,c)$ for some $(x,c) \in C(J) \times \mathbb{R}$ and $\lambda \in (0,1)$. Then

$$x(t) = \lambda\big(c + I^{\alpha-1}\mathcal{Q}x(t)\big), \ \ t \in J, \tag{3.1}$$

$$c(\lambda - 1) = \lambda I^{\alpha-1}\mathcal{Q}x(t)\big|_{t=T}. \tag{3.2}$$

It follows from (2.1) and (3.1) that

$$x'(t) = \lambda\frac{\mathrm{d}}{\mathrm{d}t}\, I^{\alpha-1}\mathcal{Q}x(t) = \lambda I^{\alpha-1}\big(\mathcal{P}x(t)\mathcal{Q}x(t) + \mathcal{F}x(t)\big), \ \ t \in J, \tag{3.3}$$

and $x'(0) = 0$. We claim that

$$|x(0)| < D, \tag{3.4}$$

where D is from (H_2). Suppose $x(0) \geq D$. Then $\mathcal{F}x(t)|_{t=0} = f(0,x(0)) > 0$ by (H_2), and therefore $\mathcal{F}x > 0$ on $[0,\rho]$ for some $\rho \in (0,T]$. Since $\mathcal{P}x(t) = p(t,x(t)) \geq 0$ on J by the assumption, we have $\mathcal{Q}x > 0$ on $(0,\rho]$ and then (see (3.3)) $x' > 0$ on this interval. Thus x is increasing on $[0,\rho]$ and so $x > D$ on $(0,\rho]$. Analysis similar to the above interval $[0,\rho]$ shows that $x \geq D$ on J. Hence $\mathcal{F}x > 0$ on J and therefore $\lambda I^{\alpha-1}\mathcal{Q}x(t)|_{t=T} > 0$ contrary to (3.2) since $c(\lambda - 1) < 0$. We have proved $x(0) < D$. Similarly we can prove $x(0) > -D$. Consequently, estimate (3.4) is valid.

Since (see (3.1)) $x(0) = \lambda c$, we have

$$x(t) = x(0) + \lambda I^{\alpha-1}\mathcal{Q}x(t), \ \ t \in J.$$

Now by applying (3.4), (H_1) and (H_3), some calculations give

$$|x(t)| \leq L_1 + L_2\int_0^t |x(s)|\,\mathrm{d}s, \ \ t \in J,$$

where L_1, L_2 are positive constants independent of λ. By the Gronwall–Bellman lemma, $\|x\| \leq L_1e^{L_2T}$.

In order to give the bound for c, two cases if $\lambda \in (0,1/2]$ or $\lambda \in (1/2,1)$ are discussed. $\square$

Example 3.1. Let $k > 0$, $\rho \in (0,1)$, $q \in C(J)$ and $w, r \in C(J \times \mathbb{R})$ be bounded, $|r(t,x)| \le P$ for $(t,x) \in J \times \mathbb{R}$. Then the function

$$f(t,x) = r(t,x) + q(t)|x|^\rho + kx$$

satisfies condition (H_3) for $A = P + \|q\|$, $B = k + \|q\|$. Since

$$\lim_{x\to\pm\infty} \frac{P + \|q\||x|^\rho}{x} = 0,$$

there exists $D > 0$ such that

$$\frac{P + \|q\||x|^\rho}{x} > -k \text{ for } x \le -D, \quad \frac{P + \|q\|x^\rho}{x} < k \text{ for } x \ge D.$$

Hence f satisfies condition (H_2). By Theorem 3.1, there exists a solution of problem

$$\begin{gathered} {}^c\!D^\alpha x - |w(t,x)|\, {}^c\!D^{\alpha-1} x = r(t,x) + q(t)|x|^\rho + kx, \\ x(0) = x(T), \quad x'(0) = 0. \end{gathered}$$

4 Uniqueness results

In this section we assume that the function $p(t,x)$ in equation (1.1) is independent of the variable x, that is, $p(t,x) = p(t)$. Hence we discuss the fractional differential equation

$$ {}^c\!D^\alpha x(t) - p(t)\, {}^c\!D^{\alpha-1} x(t) = f(t, x(t)), \tag{4.1}$$

where $p \in C(J)$. According to Lemma 2.2, x is a solution of problem (4.1), (1.2) if and only if $x \in C(J)$,

$$x(t) = x(0) + I^{\alpha-1}\mathcal{Q}x(t) \text{ for } t \in J \text{ and } x(0) = x(T),$$

where

$$\mathcal{Q}x(t) = \int_0^t \mathcal{F}x(s) \exp\left(\int_s^t p(\xi)\,\mathrm{d}\xi\right) \mathrm{d}s.$$

Let $\mathcal{A}$ be the set of all solutions to problem (4.1), (1.2). Under conditions (H_2), (H_3) and $p \ge 0$ on J, $\mathcal{A} \ne \varnothing$ and $|x(0)| < D$ for $x \in \mathcal{A}$ by Theorem 3.1. We are interested in the structure of the set $\mathcal{A}$, especially when $\mathcal{A}$ is a singleton set, that is, when problem (4.1), (1.2) has a unique solution.

Lemma 4.1. *Let $p \ge 0$ on J and let (H_2), (H_3),*

(H_4) for each $t \in J$, $f(t,x)$ is increasing in the variable x on $\mathbb{R}$

hold. Then $u(0) = v(0)$ for $u, v \in \mathcal{A}$.

The following theorem says that if $u, v \in \mathcal{A}$ and $u \ne v$, then the function $u - v$ vanishes at points t_n of a sequence $\{t_n\} \subset (0,T)$.

Theorem 4.1. *Let (H_2)–(H_4) hold and let $p \ge 0$ on J. Let $u, v \in \mathcal{A}$ and $u \ne v$. Then there exists a decreasing sequence $\{t_n\} \subset (0,T)$, $\lim_{n\to\infty} t_n = 0$, such that*

$$u(t_n) - v(t_n) = 0 \text{ for } n \in \mathbb{N}.$$

We are now in the position to give the conditions for the existence of a unique solution to problem (4.1), (1.2).

Theorem 4.2. *Let* $p \geq 0$ *on* J *and let* (H_2)–(H_4),

(H_5) f *satisfies the local Lipschitz condition on* $J \times \mathbb{R}$*, that is, for each* $S > 0$ *there is* $L = L(S) > 0$ *such that*

$$|f(t,x_1) - f(t,x_2)| \leq L|x_1 - x_2| \ \text{ for } \ t \in J, \ \ x_1, x_2 \in [-S, S]$$

hold. Then problem (4.1), (1.2) *has a unique solution.*

Example 4.1. Let $k > 0$, $\rho \in (0,1)$, $q, r \in C(J)$ and $f(t,x) = r(t) + |x|^\rho \arctan x + kx$. Then f satisfies conditions (H_2) and (H_3) for $D = \|r\|/k$ and $A = \|r\| + \pi/2$, $B = k + \pi/2$. Since the function $\phi(x) = |x|^\rho \arctan x + kx$ has continuous derivative on $\mathbb{R}$, $\frac{\partial f}{\partial x} \in C(J \times \mathbb{R})$, and therefore f satisfies condition (H_5). Clearly, f satisfies condition (H_4). Consequently, by Theorem 4.2, there exists a unique solution of problem

$$\begin{gathered} {}^c\!D^\alpha x - |q(t)|\,{}^c\!D^{\alpha-1} x = r(t) + |x|^\rho \arctan x + kx, \\ x(0) = x(T), \quad x'(0) = 0. \end{gathered}$$

References

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