A System of Singularly Perturbed Differential Equations with an Unstable Turning Point of the First Kind

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Let us consider the system of differential equations with turning point:

$$\varepsilon Y'(x,\varepsilon) - A(x,\varepsilon)Y(x,\varepsilon) = H(x), \tag{0.1}$$

where

$$A(x,\varepsilon) = A_0(x) + \varepsilon A_1(x)$$

is a known matrix, where

$$\mathbf{A_0}(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -b(x) & -a(x) & 0 \end{pmatrix}, \quad \mathbf{A_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

when $\varepsilon \to 0$, $x \in [-l,0]$, $Y(x,\varepsilon) \equiv Y_k(x,\varepsilon) = colomn(y_1(x,\varepsilon), y_2(x,\varepsilon), y_3(x,\varepsilon))$ is an unknown vector function, $H(x) = colomn(0,0,h(x))$ is a given vector function.

Needs of modern physics, mathematics, biology and their applied fields require us solving problems of a more complex nature, i.e. research behavior of the function in asymptotic models, which are reduced to problems (0.1).

Let us investigate the problem of constructing uniform asymptotics of solutions of a singularly perturbed system (0.1) for which the conditions are fulfilled:

S1. $A_0(x)$, $H(x) \in C^\infty[-l,0]$.

S2. $a(x) = x\widetilde{a}(x)$, $\widetilde{a}(x) < 0$, $b(x) \neq 0$.

This case has the following feature: the turning point is unstable [1] and the construction of asymptotics requires a separate technique, since the results of previous studies cannot be simply extended to this case.

Conducted research in [1] showed that for construction of uniform asymptotic under conditions **S2**, i.e. when $\widetilde{a}(x) < 0$, $b(x) > 0$ when $x \in [-l,0]$, the second form must be used the Airy equation, the solutions of which are the so-called Airy–Langer functions: Ai(t) and Bi(t).

$$U''(t) - tU(t) = 0.$$

That is, in this case, the model operator for a homogeneous system is the Airy model operator. And to construct the asymptotics of the solution of a heterogeneous system, we will use an essentially special function $\nu(t)$

$$U''(t) - tU(t) = \pi^{-1}.$$

Some aspects of problem (0.1) we studied in [3]. In [4], a developed algorithm for constructing uniform asymptotics of solutions to systems of singularly perturbed differential equations is proposed. In [2] constructive conditions for the existence of the asymptotics of the solution of the system of singularly perturbed differential equations of the fourth order with a differential turning point are established, and the algorithm for constructing the corresponding solution is proposed.

The characteristic equation that corresponds to the SP system (0.1) is as follows:

$$|A_0(x)-\lambda E| = \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ -b(x) & -a(x) & -\lambda \end{vmatrix} = -\lambda^3 - x\widetilde{a}(x)\lambda = 0.$$

The roots of this equation are

$$\lambda_1 = 0, \quad \lambda_{2,3} = \pm\sqrt{x\widetilde{a}(x)}\,.$$

The purpose of this work is to construct a uniform asymptotic solution with an unstable turning point of the first kind.

1 Regularization of singularly perturbed systems of differential equations

In order to save all essential singular functions, that appear in the solution of system (0.1) due to the special point $\varepsilon = 0$, a regularizing variable is introduced $t = \varepsilon^{-p}\cdot\varphi(x)$, where exponent p and regularizing function $\varphi(x)$ are to be determined.

Instead of $Y_k(x,\varepsilon)$ function, $\widetilde{Y}_k(x,t,\varepsilon)$ transformation function will be studied, also the transformation will be performed in such a way that the following identity is true

$$\widetilde{Y}(x,t,\varepsilon)\big|_{t=\varepsilon^{-p}\varphi(x)} \equiv Y(x,\varepsilon),$$

which is the necessary condition for suggested method.

The vector equation (0.1) can be written as

$$\widetilde{L}_\varepsilon \widetilde{Y}_k(x,t,\varepsilon) \equiv \mu\varphi'\,\frac{\partial\widetilde{y}(x,t,\varepsilon)}{\partial t} + \mu^3\,\frac{\partial\widetilde{y}(x,t,\varepsilon)}{\partial x} - A(x,\varepsilon)\widetilde{Y}_k(x,t,\varepsilon) = H(x). \tag{1.1}$$

Asymptotic forms of solutions for equation (1.1) are constructed in the form of the series

$$\widetilde{Y}_k(x,t,\varepsilon) = \sum_{i=1}^{2} D_i(x,t,\varepsilon) + f(x,\varepsilon)\nu(t) + \varepsilon^\gamma g(x,\varepsilon)\nu'(t) + \omega(x,\varepsilon),$$

$$\sum_{i=1}^{2} D_i(x,t,\varepsilon) = \begin{pmatrix} \varepsilon^{s1}\alpha_{k1}(x,\varepsilon) \\ \varepsilon^{s2}\alpha_{k2}(x,\varepsilon) \\ \varepsilon^{s3}\alpha_{k3}(x,\varepsilon) \end{pmatrix} U_i(t) + \varepsilon^\gamma \begin{pmatrix} \varepsilon^{k1}\beta_{k1}(x,\varepsilon) \\ \varepsilon^{k2}\beta_{k2}(x,\varepsilon) \\ \varepsilon^{k3}\beta_{k3}(x,\varepsilon) \end{pmatrix} {U_i}'(t),$$

where $U_1(t)$, $U_2(t)$ are the Airy–Langer functions [1] and $\alpha_{ik}(x,\varepsilon)$, $\beta_{ik}(x,\varepsilon)$, $f_k(x,\varepsilon)$, $g_k(x,\varepsilon)$, $\omega_k(x,\varepsilon)$, $k = 1,2,3$ are analytic functions with reference to a small parameter and are infinitely differentiable functions of variable $x \in [-l;0]$ which are still to be determined.

For convenience, we introduce the notation $U_1(t) \equiv \mathrm{Ai}(t)$, $U_2(t) \equiv \mathrm{Bi}(t)$.

First of all, the analysis how transformation operator $\widetilde{L}_\varepsilon$ operates on vector function $D_k(x,t,\varepsilon)$ will be performed, and then the obtained result will be utilized in the homogeneous transformation equation (0.1). Then, after equating corresponding coefficients of essential singular functions $U_k(t)$, $k = 1,2$ and their derivatives two following vector equations are obtained:

$$U_i'(t) : \varepsilon^{1-p}\alpha_{ik}(x,\varepsilon)\varphi'(x) - \varepsilon^\gamma[A_0(x)+\varepsilon A_1]\beta_{ik}(x,\varepsilon) = -\varepsilon^{1+\gamma}\beta_{ik}'(x,\varepsilon), \tag{1.2}$$

$$U_i(t) : -\varepsilon^{1+\gamma-2p}\beta_{ik}(x,\varepsilon)\varphi(x)\varphi'(x) - [A_0(x)+\varepsilon A_1]\alpha_{ik}(x,\varepsilon) = -\varepsilon\alpha_{ik}'(x,\varepsilon). \tag{1.3}$$

2 Construction of formal solutions of a homogeneous transformation system

The unknown coefficients of the vector equations (1.2) and (1.3) are sought as following vector function series ($i=1,2$):

$$\alpha_{ik}(x,\varepsilon)=\sum_{r=0}^{+\infty}\mu^r\alpha_{ikr}(x),\quad \beta_{ik}(x,\varepsilon)=\sum_{r=0}^{+\infty}\mu^r\beta_{ikr}(x).$$

To determine vector function components

$$\alpha_{ikr}=colomn\big(\alpha_{i1r}(x),\alpha_{i2r}(x),\alpha_{i3r}(x)\big)$$

and

$$\beta_{ikr}(x)=colomn\big(\beta_{i1r}(x),\beta_{i2r}(x),\beta_{i3r}(x)\big),$$

the following recurrent systems of equations are obtained:

$$\begin{aligned}&\Phi(x)Z_{k0}(x)=0,\ \ r=0,1,2,\\&\Phi(x)Z_{kr}(x)=FZ_{k(r-3)}(x),\ \ r\geq 3.\end{aligned}$$

At the moment, the regularizing function has not yet been defined; therefore, it will be defined as a solution of the problem

$$\varphi(x)\varphi'^2(x)=-a(x)\equiv -x\widetilde{a},\quad \varphi(0)=0,$$

which is the following function

$$\varphi(x)=\bigg(\frac{3}{2}\int\limits_0^x\sqrt{-x\widetilde{a}(x)}\,dx\bigg)^{\frac{2}{3}}.$$

The regularizing function of such kind has been considered in [1,5].

Due to such a choice of the regularizing variable $\varphi(x)$ there is a nontrivial solution of the homogeneous system $\Phi(x)Z_{kr}(x)=0$, $r=0,1,2$, that is

$$Z_{k0}(x)=colomn\Big(0,\frac{1}{\varphi'(x)}\,\beta_{i30}(x),-\varphi(x)\varphi'(x)\beta_{i20}(x),0,\beta_{i20}(x),\beta_{i30}(x)\Big),$$

where $\beta_{ikr}(x)$ ($i=1,2$; $k=2,3$) are arbitrary up to some point and sufficiently smooth function at $x\in[-l;0]$.

Solving systems of recurrent equations at the third step, i.e., when $r=3$, and taking into account that the functions are arbitrary, $\beta_{is0}(x)=\beta^0_{is0}\cdot\hat{\beta}_{is0}(x)$ ($i=1,2$; $s=2,3$), where $\beta^0_{is0}(x)$ are arbitrary constants, $\hat{\beta}_{is0}(x)$ is a partial and sufficiently smooth for all $x\in[-l;0]$ solutions of homogeneous equations. This definition of vector functions $Z_{ik0}(x)$ implies that there are following solutions of inhomogeneous systems of the algebraic equations (1.2) and (1.3):

$$\begin{gathered}Z_{k3}(x)=colomn(z_{i13},z_{i23},z_{i33},z_{i43},z_{i53},z_{i63}),\\ z_{i13}=\frac{1}{\varphi'(x)}\,\beta_{i20}(x),\ \ z_{i23}=\frac{-\beta'_{i20}(x)+\beta_{i33}(x)}{\varphi'(x)}\,,\\ z_{i33}=\frac{-\beta'_{i30}(x)-a(x)\beta_{i23}(x)-b(x)(\varphi(x))^{-1}(\varphi'(x))^{-2}\beta_{i30}}{\varphi'(x)}\,,\\ z_{i43}=(\varphi(x))^{-1}(\varphi'(x))^{-2}\beta_{i20}(x),\ \ z_{i53}=\beta_{i21}(x),\ \ z_{i63}=\beta_{i31}(x),\end{gathered}$$

where $\beta_{i21}(x)$ and $\beta_{i31}(x)$ are arbitrary up to some point and sufficiently smooth functions for all $x \in [-l; 0]$.

Thus, gradual solving of systems of equations (1.2) and (1.3) gives two formal solutions of the transformation vector equation (0.1)

$$D_{ik}(x, \varepsilon^{-\frac{2}{3}}\varphi(x), \varepsilon) = \sum_{r=0}^{\infty} \varepsilon^r \Big[\alpha_{ikr}(x) U_i(\varepsilon^{-\frac{2}{3}}\varphi(x)) + \varepsilon^{\frac{1}{3}} \beta_{ikr}(x, \varepsilon) U_i'(\varepsilon^{-\frac{2}{3}}\varphi(x))\Big]. \tag{2.1}$$

The third formal solution of the homogeneous vector equation (0.1) is then constructed as a series

$$\omega(x, \varepsilon) \equiv \sum_{r=0}^{\infty} \varepsilon^r \omega_r(x) \equiv colomn\Big(\sum_{r=0}^{\infty} \varepsilon^r \omega_{1r}(x), \sum_{r=0}^{\infty} \varepsilon^r \omega_{2r}(x), \sum_{r=0}^{\infty} \varepsilon^r \omega_{3r}(x)\Big).$$

3 Construction of formal partial solutions

Similarly to the previous steps, in order to construct asymptotic forms of partial solutions of the inhomogeneous transformation vector equation (0.1), let us analyze how transformation operator operates on an element from the space of non-resonant solutions:

$$\begin{aligned}
&\widetilde{L}_\varepsilon\big(f_k(x, \varepsilon)\nu(t) + \mu g_k(x, \varepsilon)\nu'(t) + \omega_k(x, \varepsilon)\big) \\
&\quad = \mu f_k(x, \varepsilon)\varphi'(x)\nu(t) + g_k(x, \varepsilon)\varphi'(x)\varphi(x)\nu(t) - A(x, \varepsilon) f_k(x, \varepsilon)\nu(t) - \mu A(x, \varepsilon) g_k(x, \varepsilon)\nu'(t) \\
&\qquad + \mu^3 f_k'(x)\nu(t) + \mu^4 g_k'(x)\nu'(t) + \mu^2\varphi'(x) g_k(x)\pi^{-1} + \mu^3\omega'(x) - A(x, \varepsilon)\omega_k(x) = H(x).
\end{aligned}$$

In order to have smooth solutions of the systems, the asymptotic forms of the solutions are constructed as series

$$f_k(x, \varepsilon) = \sum_{r=-2}^{+\infty} \mu^r f_r(x), \quad g_k(x, \varepsilon) = \sum_{r=-2}^{+\infty} \mu^r g_r(x), \quad \overline{\omega}(x, \varepsilon) = \sum_{r=0}^{+\infty} \mu^r \overline{\omega}_r(x).$$

Therefore, the partial solution of the transformation vector equation (0.1) is then defined as the series

$$\widetilde{Y}_k^{part.}(x, t, \varepsilon) = \sum_{r=-2}^{\infty} \mu^r \big[f_{kr}(x)\nu(t) + \mu g_{kr}(x)\nu'(t)\big] + \sum_{r=0}^{\infty} \mu^r \overline{\omega}_{kr}(x). \tag{3.1}$$

4 Conclusions

Therefore, we constructed a uniform asymptotic solution for a system of singularly perturbed differential equations with an unstable turning point (0.1) in the form (2.1) and (3.1).

References

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