Definition and Some Properties of Measures of Stability and Instability of a Differential System

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For a given $n \in \mathbb{N}$ and zero neighborhood $G \subset \mathbb{R}^n$, we consider the differential system

$$\dot{x} = f(t,x), \quad f(t,0) \equiv 0, \ \ t \in \mathbb{R}_+ \equiv [0,+\infty), \ \ x \in G, \tag{1}$$

where $f, f'_x \in C(\mathbb{R}_+, G)$. Let's put

$$B_\delta \equiv \{x_0 \in \mathbb{R}^n \mid \ 0 < |x_0| < \delta\}, \quad \Delta \equiv \sup\{\delta \mid \ B_\delta \subset G\},$$

and denote by $x(\,\cdot\,, x_0)$ a non-extendable solution of system (1) with the initial value $x(0,x_0) = x_0$.

The differential system (1) is completely deterministic, however, it is possible to give a natural stochastic meaning to its measures of stability $\mu_\varkappa(f)$ or instability $\nu_\varkappa(f)$ [1,2]. They allow us to estimate from below the possibility or impossibility of randomly selecting the initial value x_0 of perturbed solution $x(\,\cdot\,, x_0)$, arbitrarily close to zero, so that its graph falls into a given tube of the zero solution in any of the following senses [3,4]:

(a) immediately on the entire time semi-axis (the Lyapunov stability for $\varkappa = \lambda$);

(b) at least episodically, but at arbitrarily late points in time (the Perron stability for $\varkappa = \pi$);

(c) at least from some moment, but then forever (the upper-limit stability for $\varkappa = \sigma$).

The forerunners of the described measures were the recent concepts of almost stability and almost complete instability [5], which provide the corresponding properties of solutions with a full measure.

Definition 1. We will say that system (1) has the following property of the *Lyapunov, Perron* or *upper-limit* type:

(a) *stability* (*almost stability*) if for any $\varepsilon > 0$ there exists $\delta \in (0,\Delta)$ such that any (respectively, almost any in the sense of the Lebesgue measure) initial value $x_0 \in B_\delta$ satisfies the corresponding requirement

$$\sup_{t\in\mathbb{R}_+} |x(t,x_0)| < \varepsilon, \quad \varliminf_{t\to+\infty} |x(t,x_0)| < \varepsilon, \quad \varlimsup_{t\to+\infty} |x(t,x_0)| < \varepsilon; \tag{2}$$

(b) *complete instability* (*almost complete instability*) if there exist $\varepsilon > 0$ and $\delta \in (0,\Delta)$ such that any (respectively, almost any) initial value $x_0 \in B_\delta$ does not satisfy the corresponding requirement (2) (which is considered to be unfulfilled by definition, in particular, when the solution $x(\,\cdot\,, x_0)$ is not defined on the entire ray $\mathbb{R}_+$).

Definition 2. For system (1), the number

$$\mu_\varkappa(f) \in [0,1], \quad \varkappa = \lambda, \pi, \sigma,$$

is called, respectively, *the Lyapunov, Perron* and *upper-limit measure of stability*, if system (1):

(a) for each $\mu < \mu_\varkappa(f)$ is *μ-stable*, i.e. for any $\varepsilon > 0$ there exists $\delta_\varepsilon \in (0, \Delta)$ such that for every $\delta \in (0, \delta_\varepsilon)$ all values $x_0 \in B_\delta$, satisfying the corresponding requirement (2), form a subset, whose relative measure (in the Lebesgue sense) in B_δ is

$$\mathrm{M}_\varkappa(f, \varepsilon, \delta) \geq \mu;$$

(b) for each $\mu > \mu_\varkappa(f)$ is not μ-stable.

Definition 3. For system (1), the number

$$\nu_\varkappa(f) \in [0, 1], \quad \varkappa = \lambda, \pi, \sigma,$$

is called, respectively, *the Lyapunov, Perron* and *upper-limit measure of instability*, if system (1):

(a) for each $\nu < \nu_\varkappa(f)$ is *ν-unstable*, i.e. for any $\varepsilon > 0$ there exists $\delta_\varepsilon \in (0, \Delta)$ such that for every $\delta \in (0, \delta_\varepsilon)$ all values $x_0 \in B_\delta$, unsatisfying the corresponding requirement (2), form a subset, whose relative measure (in the Lebesgue sense) in B_δ is

$$\mathrm{N}_\varkappa(f, \varepsilon, \delta) \geq \nu;$$

(b) for each $\nu > \nu_\varkappa(f)$ is not ν-unstable.

The correctness of Definitions 2 and 3 is justified by the following theorems.

Theorem 1. *For any system* (1)*, any $\varepsilon > 0$ and each of the requirements* (2)*, the sets of all points $x_0 \in G$, both satisfying this requirement and not satisfying it, are measurable.*

Theorem 2. *For any system* (1) *the set of all values $\mu \in [0, 1]$ for which it is Lyapunov, Perron or upper-limit μ-stable, as well as all values $\nu \in [0, 1]$, for which it is ν-unstable, obviously contains the point 0 and represents an interval, possibly degenerate to this point.*

The following two theorems offer specific formulas for measures of stability and instability and define a set of basic relations linking various measures.

Theorem 3. *For each system* (1)*, the entire six of its Lyapunov, Perron and upper-limit measures of stability or instability are uniquely defined, which are respectively given by the formulas*

$$\mu_\varkappa(f) = \lim_{\varepsilon \to +0} \varliminf_{\delta \to +0} \mathrm{M}_\varkappa(f, \varepsilon, \delta), \quad \nu_\varkappa(f) = \lim_{\varepsilon \to +0} \varliminf_{\delta \to +0} \mathrm{N}_\varkappa(f, \varepsilon, \delta), \tag{3}$$

where the limits at $\varepsilon \to +0$ can be replaced by the lower or, respectively, upper exact bound on $\varepsilon > 0$.

Theorem 4. *For any system* (1) *the inequalities are satisfied*

$$0 \leqslant \mu_\lambda(f) \leqslant \mu_\sigma(f) \leqslant \mu_\pi(f) \leqslant 1, \quad 0 \leqslant \nu_\pi(f) \leqslant \nu_\sigma(f) \leqslant \nu_\lambda(f) \leqslant 1, \tag{4}$$

$$0 \leqslant \mu_\varkappa(f) + \nu_\varkappa(f) \leqslant 1. \tag{5}$$

Almost stability and almost complete instability are naturally associated with single values of the corresponding measures, but this logical connection turns out to be only one-way.

Theorem 5. *System* (1) *has almost stability or almost complete instability (of some type) if and only if it is 1-stable or, accordingly,* 1*-unstable (of that type), and then its measures of stability and instability (of the same type) are equal to* 1 *and* 0 *or, respectively, vice versa.*

Theorem 6. *For $n = 2$, there are two autonomous systems of the form* (1)*, which have neither almost stability nor almost complete instability of any of the three types: one of them has measures of stability and instability of all three types equal to* 1 *and* 0*, respectively, and the other is the opposite.*

In the case of a linear system, the Lyapunov and upper-limit measures can only take their extreme values, which are obviously also realized on the Perron measures – this is what the following two theorems establish.

Theorem 7. *For any linear system* (1)*, only the following two situations are possible, and in formulas* (3) *for all measures of stability and instability mentioned in them, the lower limits for $\delta \to +0$ are exact:*

(a) *either the relations are satisfied*

$$\mu_\lambda(f) = \mu_\sigma(f) = \mu_\pi(f) = 1 > 0 = \nu_\pi(f) = \nu_\sigma(f) = \nu_\lambda(f)$$

and system (1) *has stability of all three types;*

(b) *either the relations are satisfied*

$$\mu_\lambda(f) = \mu_\sigma(f) = 0 < 1 = \nu_\sigma(f) = \nu_\lambda(f)$$

and system (1) *has the Lyapunov and upper-limit almost complete (possibly even complete) instability.*

In addition, in the linear case, the upper-limit complete instability follows from the Lyapunov one, but the Perron instability does not follow, and not to any extent.

Theorem 8. *For any $n \in \mathbb{N}$, each of the situations listed in Theorem 7 is realized on some limited scalar linear system of the form* (1)*, and the second situation is realized on at least two systems: one of them is autonomous and has the Perron complete instability, i.e.*

$$\mu_\pi(f) = 0 < 1 = \nu_\pi(f),$$

and the other – the Perron stability, i.e.

$$\mu_\pi(f) = 1 > 0 = \nu_\pi(f).$$

The set of all possible sets of different measures of stability and instability of one-dimensional systems is finite.

Theorem 9. *For $n = 1$, the measures of stability and instability of any system* (1) *satisfy the relations*

$$\mu_\lambda(f) = \mu_\sigma(f) \leqslant \mu_\pi(f), \quad \nu_\pi(f) \leqslant \nu_\sigma(f) = \nu_\lambda(f), \tag{6}$$

$$\mu_\varkappa(f), \nu_\varkappa(f) \in \{0,\ 1/2,\ 1\}, \quad \mu_\varkappa(f) + \nu_\varkappa(f) = 1, \quad \varkappa = \lambda, \pi, \sigma. \tag{7}$$

Theorem 10. *For $n = 1$, both inequalities in chains* (6) *for some limited linear system* (1) *are strict, and the cases of all equalities in these chains for each pair of measures of stability and instability specified by conditions* (7) *are implemented on some autonomous systems* (1)*.*

Theorem 6 simultaneously confirms the realizability of both zero and one values by all measures of stability or instability for two-dimensional autonomous systems. Moreover, for such systems the set of implementable sets of all measures turns out to be quite rich.

Theorem 11. *For $n = 2$, for each individual non-strict inequality in chains* (4) *and* (5) *there are two autonomous systems of the form* (1)*: for one of them it turns into an equality, and for the other into a strict inequality.*

Theorem 12. *For $n = 2$, for any $r > 0$ there exists an autonomous system* (1)*, in which the measures of stability of all three types take the same positive value, as well as all measures of instability, and the ratio of these two values equals r, and the right inequality in chain* (5) *turns into equality.*

The following two theorems implement the most contrasting situations in the autonomous arbitrarily non-one-dimensional case.

Theorem 13. *For every integer $n > 1$, some autonomous system* (1) *satisfies the relations*

$$\mu_\lambda(f) = \mu_\sigma(f) = 0 < 1 = \mu_\pi(f), \quad \nu_\pi(f) = \nu_\sigma(f) = 1 > 0 = \nu_\lambda(f).$$

Theorem 14. *For every integer $n > 1$, some autonomous system* (1) *satisfies the relations*

$$\mu_\lambda(f) = 0 < 1 = \mu_\sigma(f) = \mu_\pi(f), \quad \nu_\pi(f) = 1 > 0 = \nu_\sigma(f) = \nu_\lambda(f).$$

In the one-dimensional autonomous case, two contrasting situations described in Theorems 13 and 14 are impossible.

Theorem 15. *For $n = 1$, for any autonomous system* (1) *the equalities are satisfied*

$$\mu_\lambda(f) = \mu_\sigma(f) = \mu_\pi(f), \quad \nu_\pi(f) = \nu_\sigma(f) = \nu_\lambda(f).$$

References

[1] I. N. Sergeev, Definition and properties of measures of stability and instability of zero solution of a differential system. (Russian) *Mat. Zametki* **113** (2023), no. 6, 895–904; translation in *Math. Notes* **113** (2023), no. 5-6, 831–839.

[2] I. N. Sergeev, Examples of autonomous differential systems with contrasting combinations of Lyapunov. (Russian) *Vestn. Mosk. Univ., Ser. I* **1** (2024) (in press).

[3] I. N. Sergeev, Definition and some properties of Perron stability.(Russian) *Differ. Uravn.* **55** (2019), no. 5, 636–646; translation in *Differ. Equ.* **55** (2019), no. 5, 620–630.

[4] I. N. Sergeev, Lyapunov, Perron and upper-limit stability properties of autonomous differential systems. (Russian) *Izv. Inst. Mat. Inform.* **56** (2020), 63–78.

[5] I. N. Sergeev, Massive and almost massive properties of stability and instability of differential systems. (Russian) *Differ. Uravn.* **57** (2021), no. 11, 1576–1578.