On Investigation of Differential-Algebraic Systems with Two-Point Non-Linear Boundary Conditions

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We study the differential-algebraic two-point non-linear boundary value problem on a compact interval

$$x'(t) = f(t, x(t), y(t)), \quad t \in [a, b], \tag{1}$$

$$y(t) = g(t, x(t), y(t)), \quad t \in [a, b]; \tag{2}$$

$$B(x(a), y(b)) = d, \tag{3}$$

where $f : [a, b] \times \Omega_{\varrho_0} \times \Omega_{\varrho_1} \to \mathbb{R}^p$, $g : [a, b] \times \Omega_{\varrho_0} \times \Omega_{\varrho_1} \to \mathbb{R}^q$ and $B : \Omega_{\varrho_0} \times \Omega_{\varrho_1} \to \mathbb{R}^p$ are continuous functions defined on certain bounded sets $\Omega_{\varrho_0} \subset \mathbb{R}^p$, $\Omega_{\varrho_1} \subset \mathbb{R}^q$ specified below (see (7) and (9)), $d \in \mathbb{R}^p$. We assume that the functions f, g, B satisfy the Lipschitz conditions

$$\big|f(t, u_1, v_1) - f(t, u_2, v_2)\big| \le K_1|u_1 - u_2| + K_2|v_1 - v_2|, \tag{4}$$

$$\big|g(t, u_1, v_1) - g(t, u_2, v_2)\big| \le K_3|u_1 - u_2| + K_4|v_1 - v_2|, \tag{5}$$

for $t \in [a, b]$, $\{u_1, u_2\} \subset \Omega_{\varrho_0}$, $\{v_1, v_2\} \subset \Omega_{\varrho_1}$, where K_1, K_2, K_3, K_4 are non-negative matrices of dimensions $p \times p$, $p \times q$, $q \times p$, $q \times q$ such that the spectral radii of Q and K_4 satisfy the inequalities

$$r(Q) < 1, \quad r(K_4) < 1, \tag{6}$$

where

$$Q = \frac{3}{10}\,(b - a)K$$

and K is the $(p+q)\times(p+q)$ block matrix $K=\begin{pmatrix}K_1 & K_2\\ K_3 & K_4\end{pmatrix}$. By a solution of problem (1)–(3) we understand a pair of a continuously differentiable $x:[a,b]\to\Omega_{\varrho 0}$ and continuous $y:[a,b]\to\Omega_{\varrho 1}$ functions, satisfying (1)–(3).

We show that techniques similar to those of [1, 2] can be effectively applied to the study of problem (1), (2). Note that, although condition (6) ensures that equation (2) can theoretically be solved with respect to y, it may be difficult or impossible to do this explicitly, and this is not required in our approach.

In the sequel, 1_l is the unit matrix of dimension l. For any $x=\operatorname{col}(x_1,\dots,x_l)$, $y=\operatorname{col}(y_1,\dots,y_l)$, we write $|x|=\operatorname{col}(|x_1|,\dots,|x_l|)$ and understand $x\le y$ as $x_i\le y_i$ for all $i=1,2,\dots,l$. The operations max and min for vector functions are also understood componentwise. Given any non-negative vector $\varrho\in\mathbb{R}^l$, we put

$$O_\varrho(z)=\big\{\xi\in\mathbb{R}^l:\ \ |\xi-z|\le\varrho\big\}$$

for $z\in\mathbb{R}^l$ and

$$O_\varrho(U)=\bigcup_{z\in U}O_\varrho(z)$$

for a set $U\subset\mathbb{R}^l$. The set $O_\varrho(U)$ may be called the componentwise ϱ-neighbourhood of U.

Fix certain compact convex sets $D_0\subset\mathbb{R}^p$, $D_1\subset\mathbb{R}^p$ and put

$$\Omega_{\varrho_0}=O_{\varrho_0}(D_{a,b}), \tag{7}$$

where ϱ_0 is a non-negative vector and

$$D_{a,b}=\big\{(1-\theta)z+\theta\eta:\ \ z\in D_0,\ \ \eta\in D_1,\ \ \theta\in[0,1]\big\}.$$

Choose some $\widetilde{y}\in\mathbb{R}^q$ and put

$$Y=\max_{(t,z,\eta)\in[a,b]\times D_0\times D_1}\big|g(t,x_0(t,z,\eta),\widetilde{y})-\widetilde{y}\big|,$$

where

$$x_0(t,z,\eta)=\Big(1-\frac{t-a}{b-a}\Big)z+\frac{t-a}{b-a}\,\eta,\ \ t\in[a,b], \tag{8}$$

for any $z\in D_0$, $\eta\in D_1$. Take a non-negative vector ϱ_1 and put

$$\Omega_{\varrho_1}=O_{\varrho_1}(\overline{y}_1), \tag{9}$$

where

$$\overline{y}_1=\max_{(t,z,\eta)\in[a,b]\times D_0\times D_1}|g(t,x_0(t,z,\eta),\widetilde{y})|.$$

We assume in what follows that the non-negative vectors ϱ_0, ϱ_1 can be chosen so that

$$\varrho_0\ge\frac{1}{2}\,(b-a)\delta_{\varrho_0,\varrho_1}(f), \tag{10}$$

$$\varrho_1\ge(1_q-K_4)^{-1}\Big(\frac{1}{2}\,(b-a)K_3\delta_{\varrho_0,\varrho_1}(f)+K_4Y\Big)+Y, \tag{11}$$

where

$$\delta_{\varrho_0,\varrho_1}(f)=\frac{1}{2}\,\Big(\max_{(t,x,y)\in[a,b]\times\Omega_{\varrho_0}\times\Omega_{\varrho_1}}f(t,x,y)-\min_{(t,x,y)\in[a,b]\times\Omega_{\varrho_0}\times\Omega_{\varrho_1}}f(t,x,y)\Big).$$

Together with inequalities (6), conditions (10), (11) may be regarded as smallness conditions on the functions f and g in a neighbourhood of sets D_0 and D_1. When some of these conditions are violated, one can apply a technique from [3] in order to construct convergent iterations.

Instead of the original boundary value problem (1)–(3), consider the family of auxiliary two-point boundary value problems

$$\begin{aligned} x'(t) &= f(t, x(t), y(t)), \quad t \in [a, b], \\ y(t) &= g(t, x(t), y(t)), \quad t \in [a, b]; \end{aligned} \tag{12}$$

$$x(a) = z, \quad x(b) = \eta, \tag{13}$$

where $z \in D_0$ and $\eta \in D_1$ are free parameters. We focus on continuously differentiable $x : [a, b] \to \Omega_{\varrho_0}$ and continuous $y : [a, b] \to \Omega_{\varrho_1}$ solutions of problem (12), (13) with values $x(a) \in D_0$ and $x(b) \in D_1$. As it will be indicated below, one can then go back to the original problem by choosing the values of z and η appropriately.

In relation to the two-point boundary value problem (12), (13), introduce the sequences of functions

$$\begin{aligned} x_{m+1}(t, z, \eta) = z + \int_a^t f\big(s, x_m(s, z, \eta), y_m(s, z, \eta)\big)\, ds \\ - \frac{t-a}{b-a} \int_a^b f\big(s, x_m(s, z, \eta), y_m(s, z, \eta)\big)\, ds + \frac{t-a}{b-a}\, (\eta - z), \end{aligned} \tag{14}$$

$$y_{m+1}(t, z, \eta) = g(t, x_m(t, z, \eta), y_m(t, z, \eta)), \quad t \in [a, b], \quad m = 1, 2, \dots, \tag{15}$$

where

$$y_0(t, z, \eta) = \widetilde{y}, \quad t \in [a, b],$$

with a fixed value of $\widetilde{y}$ and the function x_0 given by (8).

For $1 \le i_1 < i_2 \le n$, let J_{i_1,i_2} be the $(i_2 - i_1 + 1) \times n$ block matrix with the unit matrix of dimension $i_2 - i_1 + 1$ placed starting from the i_1th column, that is

$$J_{i_1,i_2} = \begin{pmatrix} 0 & 1_{i_2 - i_1 + 1} & 0 \end{pmatrix}, \tag{16}$$

where the symbols 0 stand for the zero blocks of appropriate dimensions.

Theorem 1. *Let conditions* (4)–(6) *and* (10), (11) *be fulfilled. Then, for all fixed* $z \in D_0$ *and* $\eta \in D_1$*:*

1) *The functions of sequence* (14) *have range in* Ω_{ϱ_0}*, satisfy the two-point conditions* (13)*, and the limit*

$$x_\infty(t, z, \eta) = \lim_{m \to \infty} x_m(t, z, \eta)$$

exists uniformly on $[a, b] \times D_0 \times D_1$*. The function* $x_\infty(\,\cdot\,, z, \eta)$ *satisfies conditions* (13) *and is continuously differentiable.*

2) *The functions of sequence* (15) *have range in* Ω_{ϱ_1} *and converge to a continuous limit function*

$$y_\infty(t, z, \eta) = \lim_{m \to \infty} y_m(t, z, \eta)$$

uniformly on $[a, b] \times D_0 \times D_1$*.*

3) *The functions $x=x_\infty(\,\cdot\,,z,\eta)$, $y=y_\infty(\,\cdot\,,z,\eta)$ satisfy equation* (2) *and the Cauchy problem with a constant forcing term:*

$$x'(t)=f(t,x(t),y(t))+\frac{1}{b-a}\,\Delta(z,\eta),\ \ t\in[a,b];\ \ x(a)=z, \tag{17}$$

where $\Delta: D_0\times D_1\to\mathbb{R}^p$ is the mapping given by the formula

$$\Delta(z,\eta)=\eta-z-\int_a^b f\big(s,x_\infty(s,z,\eta),y_\infty(s,z,\eta)\big)\,ds. \tag{18}$$

Other couples of functions (x,y) having range in the set $\Omega_{\varrho_0}\times\Omega_{\varrho_1}$ and satisfying (2), (17) *do not exist.*

4) *The estimates*

$$\big|x_\infty(t,z,\eta)-x_m(t,z,\eta)\big|\le\frac{10}{9}\,\alpha_1(t)J_{1,p}K^\sigma Q^{m-\sigma}(1_{p+q}-Q)^{-1}\begin{pmatrix}\delta_{\varrho_0,\varrho_1}(f)\\ Y\end{pmatrix},$$

$$\big|y_\infty(t,z,\eta)-y_m(t,z,\eta)\big|\le\frac{10}{9}\,\alpha_1(t)J_{p+1,p+q}K^{\sigma+1}Q^{m-\sigma-1}(1_{p+q}-Q)^{-1}\begin{pmatrix}\delta_{\varrho_0,\varrho_1}(f)\\ Y\end{pmatrix}$$

hold for all $t\in[a,b]$ and m sufficiently large, where

$$\sigma=\begin{cases}0 & \text{if } b-a>\dfrac{10}{3}\,,\\[2mm] 1 & \text{if } b-a\le\dfrac{10}{3}\,.\end{cases}$$

By (16), the left multiplication of a vector column by $J_{1,p}$ (resp., $J_{p+1,p+q}$) means the selection of the components $1,2,\dots,p$ (resp., $p+1,\dots,p+q$).

For $z\in\Omega_{\varrho_0}$, $\eta\in\Omega_{\varrho_1}$, let us put

$$\Lambda(z,\eta)=B\big(x_\infty(a,z,\eta),y_\infty(b,z,\eta)\big)-d.$$

The functions Δ and Λ determine the relation of the functions $x_\infty(\,\cdot\,,z,\eta)$ and $y_\infty(\,\cdot\,,z,\eta)$ to solutions of the original problem (1)–(3).

Theorem 2. *Under the assumptions of Theorem 1, the couple of functions $(x_\infty(\,\cdot\,,z,\eta),y_\infty(\,\cdot\,,z,\eta))$ is a solution of the boundary value problem* (1)–(3) *if and only if the parameters z, η satisfy the system of $2p$ algebraic or transcendental equations*

$$\Delta(z,\eta)=0,\quad \Lambda(z,\eta)=0. \tag{19}$$

The next statement proves that the system of determining equations (19) determines all possible solutions of the original non-linear boundary value problem (1)–(3) in the regions Ω_{ϱ_0}, Ω_{ϱ_1}.

Theorem 3. *Under the assumptions of Theorem 1, the following assertions hold:*

1) *If there exist a pair of vectors $(z_*,\eta_*)\in D_0\times D_1$ satisfying the system of determining equations* (19), *then the boundary value problem* (1)–(3) *has a solution (x_*,y_*) such that $x_*([a,b])\subset\Omega_{\varrho_0}$, $y_*([a,b])\subset\Omega_{\varrho_1}$ and*

$$x_*(a)=z_*,\quad x_*(b)=\eta_*.$$

Moreover, this solution has the form

$$(x_*,y_*)=\big(x_\infty(\,\cdot\,,z_*,\eta_*),y_\infty(\,\cdot\,,z_*,\eta_*)\big).$$

2) *If problem* (1)–(3) *has a solution* (x_*, y_*) *with* $(x_*(a), x_*(b)) \in D_0 \times D_1$ *and range in* $\Omega_{\varrho_0} \times \Omega_{\varrho_1}$, *then the system of determining equations* (19) *is satisfied with* $z = x_*(a)$, $\eta = x_*(b)$.

The practical investigation of problem (1)–(3) is carried out by studying the approximate determining equations

$$\Delta_m(z, \eta) = 0, \quad \Lambda_m(z, \eta) = 0, \tag{20}$$

where

$$\Lambda_m(z, \eta) = B\big(x_m(a, z, \eta), y_m(b, z, \eta)\big) - d$$

and

$$\Delta_m(z, \eta) = \eta - z - \int_a^b f\big(s, x_m(s, z, \eta), y_m(s, z, \eta)\big)\, ds,$$

for some fixed m. Assuming that the Lipschitz condition holds for B,

$$|B(u_1, v_1) - B(u_2, v_2)| \le K_5|u_1 - u_2| + K_6|v_1 - v_2|,$$

under additional assumptions, we prove the existence of a solution of the original problem (1)–(3) by showing that the solvability of the approximate determining equations (20) in the respective region implies that of (19). A practical computation using Maple confirms the constructiveness of the proposed approach.

References

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