# The Optimal Control Problem of a System of Integro-Differential Equations on Infinite Horizon

**Lakhva Roksolana**

*Taras Shevchenko National University of Kyiv, Ukraine*
*E-mail:* roksolanalakhva@knu.ua

**Viktoriia Mogylova**

*Igor Sikorsky Kyiv Polytechnic Institute, Kyiv, Ukraine*
*E-mail:* mogylova.viktoria@gmail.com

**Tatiana Koval'chuk**

*State University of Trade and Economics, Kyiv, Ukraine*
*E-mail:* 8172@ukr.net

## Abstract

We consider the problem of optimal control for a system of integro-differential equations on the half-axis. Sufficient optimality conditions are derived in terms of the right-hand side of the system and the functions involved in the cost function. This task is distinctive in that it is analyzed up to the moment when the solution reaches the boundary of the region, which depends on the control. The proof of existence is based on the compactness approach with the identification of a minimizing sequence, followed by a limit transition in the equation and the cost function.

## 1 Problem statement

We consider the optimal control problem for a system of integro-differential equations:

$$\begin{cases} \dot{x} = f_1(t,x) + f_2(t,x)u(t) + \int\limits_0^t f_3(t,s,x)u(s)\,ds, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

with a cost function on the infinite interval:

$$J(u) = \int\limits_0^\tau e^{-\gamma t} L(t, x(t), u(t))\,dt \to \inf, \tag{1.2}$$

where $x_0 \in D$ is a fixed vector, $t \in [0,\infty)$, $x \in D$ is the phase vector, $D$ is a bounded region in $\mathbb{R}^d$, $\partial D$ is the boundary of $D$, $\tau$ is the first moment when the solution $x(t)$ reaches to $\partial D$, $u \in U \subset \mathbb{R}^m$ is the control vector, $U$ is a convex, closed set in $\mathbb{R}^m$, and $0 \in U$.

Let the following conditions be satisfied:

(A) Vector function $f_1(t,x) : [0,\infty) \times D \to \mathbb{R}^d$, matrix $f_2(t,x) : [0,\infty) \times D \to \mathbb{R}^d \times \mathbb{R}^m$, and matrix $f_3(t,s,x) : [0,\infty) \times [0,\infty) \times D \to \mathbb{R}^d \times \mathbb{R}^m$ are continuous with respect to all variables.

(B) Functions $f_1(t,x)$, $f_2(t,x)$, $f_3(t,s,x)$ satisfy the Lipschitz condition, i.e. there exists a constant $H>0$ such that for any $x_1,x_2\in D$, $t\geq 0$, and $u\in U$, the following inequalities hold:

$$\begin{gathered}|f_1(t,x_1)-f_1(t,x_2)|\leq H|x_1-x_2|,\\ \|f_2(t,x_1)-f_2(t,x_2)\|\leq H|x_1-x_2|,\\ \|f_3(t,s,x_1)-f_3(t,s,x_2)\|\leq H|x_1-x_2|,\end{gathered}$$

here we have $|\cdot|$ – vector norm, $\|\cdot\|$ – matrix norm.

The functions $L(t,x,u)$, $L_x(t,x,u)$, and $L_u(t,x,u)$ are continuous with respect to all variables and the following conditions hold:

(1) $L(t,x,u)\geq 0$ for $t\in[0,\infty)$, $x\in D$ and $u\in U$;

(2) there exist constants $C>0$ and $p\geq 2$ such that for any $t\in[0,\infty)$, $x\in D$, $u\in U$, the following inequality holds:

$$L(t,x,u)\leq C\big(1+|u|^p\big);$$

(3) there exists a constant $K>0$ such that for any $t\in[0,\infty)$, $x\in D$, $u\in U$, the next inequality holds:

$$|L_x(t,x,u)|+|L_u(t,x,u)|\leq K\big(1+|u|^{p-1}\big);$$

(4) $L(t,x,u)$ is convex with respect to $u$ for any fixed $t\in[0,\infty)$, $x\in D$.

Control $u(t)$ is considered admissible if:

(a1) $u(t)\in L_p([0,\infty])$;

(a2) $u(t)\in U$, for $t\in[0,\infty]$;

(a3) there exists a constant $C_1>0$ that is independent of $u(t)$ and the next condition holds:

$$\int\limits_0^\infty |u(t)|^p\,dt\leq C_1;$$

(a4) $|J(u)|<\infty$.

The set of admissible controls is marked as "$V$" for problem (1.1), (1.2).

For systems of ordinary differential equations, similar problems were studied in works [3], for stochastic in [4], for functional-differential systems in [1], and for impulsive systems in [2].

## 2 Main result

The main result of this paper concerns the existence of a solution of problem (1.1), (1.2). We obtained the following theorem.

**Theorem 2.1.** *Let system* (1.1) *with the quality criterion* (1.2) *satisfy conditions* (A), (B), *and* (1)–(3). *Then problem* (1.1), (1.2) *has a solution in the class of admissible controls* $V$, *i.e. there exists an optimal control* $u^*(t)$ *which minimizes the cost function* (1.2).

*Proof.* Because the cost function is a non-negative quantity, there exists a non-negative lower bound $m$ for the values of $J(u)$. Therefore, there exists a sequence of admissible controls $\{u_n(t), n \geq 1\}$ such that $J(u_n) \to m$ monotonically as $n \to \infty$.

Since $U$ is a convex and closed set, then using Mazur's lemma we obtain $u^*(t)$ almost everywhere for $t$.

For the solutions $x_n(t)$, we have the integral representation:

$$x_n(t) = x_0 + \int_0^t \left[ f_1(t, x_n(t)) + f_2(t, x_n(t))u_n(t) + \int_0^s f_3(s, \sigma, x_n(s))u(\sigma)\, d\sigma \right] dt.$$

Using the functions $x_n$, we build functions $y_n(t)$ defined on $[0, \infty)$ as follows:

$$y_n(t) = \begin{cases} x_n(t), & t \in [0, \tau_n), \\ x_n(\tau_n), & t \geq 0. \end{cases}$$

It is easy to see the set of function $y_n$ is compact in the space of continuous functions defined on $[0, T]$ for arbitrary $t > 0$. So, there exist a subsequence $\{y_{n_k}(t), n \geq 1\}$ of a sequence $\{y_n(t), n \geq 1\}$ such that $\{y_{n_k}(t), n \geq 1\}$ uniformly on the interval $[0, T]$.

Using the diagonal method, we can show that some subsequence of the sequence $\{y_{n_n}(t), n \geq 1\}$ converges pointwise to a continuous function $y^*(t)$ for any $t \in [0, \infty)$.

For convenience, we denote this subsequence again as $\{y_n(t), n \geq 1\}$ and the corresponding control sequence as $\{u_n(t), n \geq 1\}$.

Let $\tau^*$ denote the moment of the first exit of $y^*(t)$ to the boundary $\partial D$, so

$$\tau^* = \begin{cases} \inf \{ t \geq 0 : \ y^*(t) \in \partial D \}, \\ \infty, \ \text{if} \ y^*(t) \in D, \ \ \forall\, t \geq 0, \end{cases}$$

$$\tau_n = \begin{cases} \inf \{ t \geq 0 : \ y_n(t) \in \partial D \}, \\ \infty, \ \text{if} \ y_n(t) \in D, \ \ \forall\, t \geq 0. \end{cases}$$

We will show that $\tau^* \leq \lim\limits_{n \to \infty} \inf \tau_n$.

Really assume that this is not true. Then $\tau^* > \lim\limits_{n \to \infty} \inf \tau_n = \tau$. Let's consider two cases:

(1) Suppose $\tau^* < \infty$. Choose any $T_1 \in [0, \infty)$ such that $T_1 \geq \tau^*$. On the interval $[0, T_1]$, $y_n(t) \to y^*(t)$, $n \to \infty$.

By the characterization theorem of the lower limit, for any $\delta > 0$ the set $\{n \in \mathbb{N} : \ \tau_n < \tau + \delta\}$ is infinite. Choose $\delta$ such that $\tau + \delta < \tau^*$. Then, there exists a subsequence $\{\tau_{n_k}, n_k \geq 1\}$ of $\{\tau_n, n \geq 1\}$ such that $\tau_{n_k} < \tau + \delta$. Choose a moment $t_0$ such that $t_0 \in (\tau + \delta, \tau^*)$. Then $y_{n_k}(t_0) = x_{n_k}(\tau_{n_k}) \in \partial D$.

From the uniform convergence of $y_n(t)$ to $y^*(t)$ on $[0, T_1]$, we have for any $\varepsilon > 0$ that there exists $N \in \mathbb{N}$ such that for any $n_k \geq N$, the following inequality holds:

$$|y^*(t) - y_{n_k}(t)| < \varepsilon.$$

However, by choosing $\varepsilon$ such that $0 < \varepsilon < \inf\limits_{\upsilon \in \partial D} |y^*(t_0) - \upsilon|$, then for a fixed $t_0 \in (\tau + \delta, \tau^*)$, we obtain

$$|y^*(t_0) - y_{n_k}(t)| = \left| y^*(t_0) - x_{n_k}(\tau_{n_k}) \right| > \varepsilon.$$

So, we get a contradiction.

(2) Suppose $\tau^* = \infty$ and $\lim\limits_{n\to\infty} \inf \tau_n < \infty$. Similarly to the previous case, we have

$$\tau^* \leq \lim_{n\to\infty} \inf \tau_n.$$

Set $x^*(t) = y^*(t)$ for $t \in [0, \tau^*]$, in the case of finite $\tau^*$ and $x^*(t) = y^*(t)$ for $t \in [0, \infty)$ in the case of $\tau^* = \infty$.

Now, let's show that $x^*(t)$ is a solution to system (1.1), for all $t$ until it reaches the boundary, corresponding to the control $u^*(t)$.

Take any $t \in [0, \tau^*]$, for $\tau^* < \infty$, and $t \in [0, \infty)$ for $\tau^* = \infty$. Choose a sufficiently large $T \geq 0$ so that for any such $t$, $y_n(t) = x_n(t)$ for sufficiently large $n$. Since $y_n(t) \to y^*t)$ as $n \to \infty$ uniformly on $[0, T]$, $x_n(t) \to x^*(t)$ uniformly on $[0, \tau_1^*]$, where

$$\tau_1^* = \begin{cases} \inf \big\{ t \in [0, T] : \text{ if } x^*(t) \in \partial D \big\}, \\ T, \text{ if } x^*(t) \in D \setminus \partial D, \ \ \forall \, t \geq 0. \end{cases}$$

Since $x_n(t)$ is a solution to system (1.1), we have

$$\begin{gathered} x_n(t) = x_0 + \int\limits_0^t \bigg( f_1(s, x_n(s)) + f_2(s, x_n(s)) u^*(s) + \int\limits_0^s f_3(s, \sigma, x_n(s)) u^*(\sigma) \, d\sigma \bigg) \, ds \\ + \int\limits_0^t \big( f_2(s, x_n(s)) - f_2(s, x^*(s)) \big) (u_n(s) - u^*(s)) \, ds \\ + \int\limits_0^t \int\limits_0^s \big( f_3(s, \sigma, x_n(s)) - f_3(s, \sigma, x^*(s)) \big) (u_n(\sigma) - u^*(\sigma)) \, d\sigma \, ds \\ + \int\limits_0^t f_2(s, x^*(s)) (u_n(s) - u^*(s)) \, ds + \int\limits_0^t \int\limits_0^s f_3(s, \sigma, x^*(s)) (u_n(\sigma) - u^*(\sigma)) \, d\sigma \, ds. \end{gathered}$$

The convergence of each integral can be easily proven by Lebesgue's dominated convergence theorem, and the definition of weak convergence.

Then, by taking the limit as $n \to \infty$, we obtain

$$x^*(t) = x_0 + \int\limits_0^t \bigg( f_1(s, x^*(s)) + f_2(s, x^*(s)) u^*(s) + \int\limits_0^s f_3(s, \sigma, x^*(s)) u^*(\sigma) \, d\sigma \bigg) \, ds$$

for any $t \in [0, \tau_1^*]$.

So, we conclude that $x^*(t)$ is a solution to system (1.1), corresponding to the control $u^*(t)$ for $t \in [0, \tau_1^*]$.

Since the time moment $T$ is chosen arbitrarily, we have that $x^*(t)$ is a solution to system (1.1) corresponding to the control $u^*(t)$ for $t \geq 0$ until the solution reaches the boundary of the region.

As $x_n(t)$ coincides with $y_n(t)$ up to this moment, the sequences $\{x_n(t), n \geq 1\}$ converge pointwise to $x^*(t)$ for any $t \in [0, \tau_1^*]$.

Now, we prove that the control $u^*(t)$ is optimal. Consider two cases:

(1) Suppose $x^*(\tau^*) \in \partial D$. Since $L(t, x, \cdot\,)$ is convex, the following inequality holds:

$$e^{-\gamma t} \big( L(t, x^*(t), \upsilon(t)) \big) \geq e^{-\gamma t} \big( L(t, x^*(t), u^*(t)) \big) + (\upsilon(t) - u^*(t)) e^{-\gamma t} \big( L_\upsilon(t, x^*(t), u^*(t)) \big),$$

$\upsilon \in U$, $t \in [0, \tau^*]$.

Set $\upsilon = u_n(t)$. Then we easily get the following inequality

$$\lim_{n\to\infty} \int_0^{\tau^*} e^{-\gamma t} L(t, x^*(t), u_n(t))\, dt \geq \int_0^{\tau^*} e^{-\gamma t} L(t, x^*(t), u^*(t))\, dt.$$

Also, we have:

$$J(u^*) \leq \lim_{n\to\infty} \inf \int_0^{\tau^*} e^{-\gamma t} L(t, x_n(t), u_n(t))\, dt \leq \lim_{n\to\infty} J(u_n(t)).$$

Since

$$\inf_{u\in U} J(u) \leq J(u^*) \leq \lim_{n\to\infty} \inf J(u_n) = m,$$

we conclude that

$$J(u^*) = m.$$

Therefore, $u^*(t)$ is the optimal control.

(2) Now let $\tau^* = \infty$ and $x^*(t) \in D \setminus \partial D$, $t \geq 0$.

It is easy to show that the function $L(t, x^*(t), u_n(t))$ is integrable on $[0, \infty)$. Since $L(t, x, \cdot)$ is convex, the following inequality holds:

$$e^{-\gamma t} L(t, x^*(t), \upsilon(t)) \geq e^{-\gamma t} L(t, x^*(t), u^*(t)) + (\upsilon(t) - u^*(t)) e^{-\gamma t} L_\upsilon(t, x^*(t), u^*(t)),$$

$\upsilon(t) \in V$, $t \in [0, \infty)$.

Furthermore, due to weak convergence, we obtain the satisfaction of the following inequality:

$$\lim_{n\to\infty} \int_0^{\infty *} e^{-\gamma t} L(t, x^*(t), u_{n_k}(t))\, dt \geq \int_0^{\infty} e^{-\gamma t} L(t, x^*(t), u^*(t))\, dt.$$

Let's also define $J(u_n)$ as follows:

$$J(u_n) = \int_0^{\infty} e^{-\gamma t} L(t, x_n(t), u_n(t))\, dt = \int_0^{\infty} e^{-\gamma t} \big[L(t, x_n(t), u_n(t)) - L(t, x^*(t), u_n(t))\big]\, dt$$
$$+ \int_0^{\infty} e^{-\gamma t} \big[L(t, x^*(t), u_n(t)) - L(t, x^*(t), u^*(t))\big]\, dt + \int_0^{\infty} e^{-\gamma t} L(t, x^*(t), u^*(t))\, dt.$$

We obtain

$$\lim_{n\to\infty} \inf J(u_n(t)) \geq J(u^*).$$

Since

$$\inf_{u\in U} J(u) \leq J(u^*) \leq \lim_{n\to\infty} \inf J(u_n(t)) = m,$$

then

$$J(u^*) = m.$$

Thus, $u^*(t)$ is the optimal control. □

## References

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