Existence an Uniqueness of Weak Solutions of Stochastic Functional-Differential Neutral Equations in Hilbert Space

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We consider the following stochastic functional-differential neutral equation on Hilbert space:

$$d(u(t) - g(u_t)) = (f(u_t) + Au)\,dt + \sigma(u_t)\,dW(t), \quad t \geq 0, \tag{0.1}$$

$$u(t) = \phi(t), \quad t \in [-h, 0], \tag{0.2}$$

where

- $u_t = u(t + \theta)$, $\theta \in [-h, 0]$;
- A – linear operator on separable Hilbert space H;
- $W(t)$ – Q-Wiener process on separable Hilbert space K;
- $u(t)$ – state process;
- f – functional from $C([-h, 0], H)$ into H;
- σ – mapping from same space to special space of Hilbert–Smidt operators;
- $\phi : [-h, 0] \to H$ – initial condition,

while existence and uniqueness of a mild solution of the given equation (0.1), (0.2) is known, weak solutions is relatively undiscovered field.

Thus, we consider existence of weak solutions of equation (0.1), (0.2).

1 Preliminaries

Let's assume that K and H are Hilbert spaces, and V, V' is such Banach spaces that

$$V \subset H = H' \subset V'$$

is a Gelfand triple.

Let (Ω, F, P) be a complete probability space equipped with a normal filtration $\{F_t; t \geq 0\}$ generated by the Q-Wiener process W on (Ω, F, P) with the linear bounded covariance operator such that $\operatorname{tr} Q < \infty$.

We assume that there exist a complete ortonormal system e_k in K and a sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k$, $k = 1, 2, \ldots$, and

$$\sum_{k=1}^{\infty} \lambda_k < \infty.$$

The Wiener process admits the expansion $W(t)=\sum\limits_{k=1}^{\infty}\lambda_k\beta_k(t)e_k$, where $\beta_k(t)$ are real valued Brownian motions mutually independent on (Ω,F,P).

Let $U_0=Q^{\frac{1}{2}}(U)$ and $L_0^2=L_2(U_0,H)$ be the space of all Hilbert–Schmidt operators from U_0 to H with the inner product $(\Phi,\Psi)L_0^2=\operatorname{tr}[\Phi Q\Psi^*]$ and the norm $\|\Phi\|_{L_2^0}$, respectively.

$C:=C([-h,0];H)$ is the space of continuous mappings from $[-h,0]$ to H equipped with the norm $\|u\|_C=\sup\limits_{\theta\in[-h,0]}\|u(\theta)\|$, and $L_2^V:=L_2((-h,0);V)$ is the space of V-valued mappings with the norm

$$\|u\|_V^2:=\int\limits_{-h}^{0}\|u(t)\|_V^2\,dt.$$

2 Conditions on functions

To ensure existence and uniqueness of a solution, we have to impose additional conditions on functions A, f, σ, g.

Conditions on A:

(A1) Domain of A - $D(A)$ is dense in H such that $A:V\to V'$;

(A2) For any $u,v\in V$ there exist $\alpha>0$:

$$|\langle Au,v\rangle|\le\alpha\|u\|_V\|v\|_V;$$

(A3) A satisfies the coercitivity condition: $\exists\,\beta>0$, γ:

$$\langle Av,v\rangle\le-\beta\|v\|_V^2+\gamma\|v\|_V^2,\ \ \forall\,v\in V.$$

Conditions on g:

(G1) g are mapping from $C\cap L_V^2$ to H;

(G2) (Growth condition) $\exists\,K>0$:

$$\|g(\phi)\|_V^2\le K(1+\|\phi\|_{L_2^V}^2),\ \ \forall\,\phi\in L_2^V;$$

(G3) (Lipshitz condition) $\exists 1/2>L>0$:

$$\|g(\phi)-g(\psi)\|_V\le L\|\phi(t)-\psi(t)\|_V,\ \ \forall\,t\in V.$$

Composite conditions:

(C1) f is a mapping from $C\cap L_V^2$ to H, σ is a mapping from $C\cap L_V^2$ to L_2^0;

(C2) (Growth condition) There $\exists\,K>0$, $\theta\ge1$:

$$\|f(\phi)\|_V\le K\bigg(1+\Big(\int\limits_{-h}^{0}\|\phi(t)\|_V\,dt\Big)^{\theta}+\|\phi\|_V^{\theta}\bigg)$$

and

$$\|\sigma(\phi)\|_{L_2^0}^2\le K\big(1+\|\phi\|_C^2\big)$$

$\forall\,\phi\in C\cap L_V^2$.

(C3) (Coercitivity condition) There $\exists\,\beta > 0$, λ, C_1: $\forall\,\phi \in C \cap L^2_V$:

$$\langle A\phi(0), \phi(0)\rangle + \langle f(\phi), \phi(0)\rangle + \|\sigma(\phi)\|^2_{L_2^V} \le -\beta\|\phi(0)\|^2_V + \lambda\|\phi\|^2_C + C_1.$$

(C4) (Monotonicity condition) There $\exists\,\delta > 0$: $\forall\,\phi, \phi_1 \in C \cap L^2_V$:

$$2\big\langle A(\phi(0) - \phi_1(0)), \phi(0) - \phi_1(0)\big\rangle \\ + 2\big\langle f(\phi) - f(\phi_1), \phi(0) - \phi_1(0)\big\rangle + \|\sigma(\phi) - \sigma(\phi_1\|^2_{L_2^0} \le \delta\|\phi - \phi_1\|^2_C\,.$$

3 Main results

Definition. We call an F_t adapted random process $(u(t)) \in V$ weak solution for equation (0.1), (0.2) if:

(1) $u(t) = \phi(t)$, $t \in [-h, 0]$;

(2) $u \in L_2(\Omega \times [0, T], V)$;

(3) $\forall\, v \in V$, $t \in [0, T]$:

$$(u(t) - g(u_t), v) = (\phi - g(\phi), v) + \int_0^t (f(u_s) + Au, v)\, ds + \int_0^t (\sigma(u_s), v)\, dW(s).$$

Theorem (Existence and uniqueness)**.** *Suppose that conditions* (A1)–(A3)*,* (G1)–(G3) *and* (C1)–(C4) *hold, then* $\forall\phi \in C \cap L^2_V$ *equation* (0.1)*,* (0.2) *has a unique weak solution on* $[0, T]$ *such that*

$$u \in C([0, T] \times \Omega; H) \cap L_2([0, T] \times \Omega; V).$$

Moreover, the energy equation holds:

$$\|u - g(u_t)\|^2 = \|\phi - g(\phi)\|^2 \\ + \int_0^t \big\langle Au(s) + f(u_s), u(s)\big\rangle\, ds + \int_0^t \|\sigma(u_s)\|^2_{L_2^0}\, ds + \int_0^t \langle\sigma(u_s), u(s)\rangle\, dW(s).$$

Sketch of the proof:

Step 1: We consider projections of equation (0.1), (0.2) into sequence of finite-dimensional subspaces which looks as follows:

$$d\big(u^n(t) - g^n(u^n_t)\big) = \big(f^n(u^n_t) + Au^n(t)\big)\, dt + \sigma^n(u^n_t)\, dW^n(t), \tag{3.1}$$

$$u^n(t) = \phi^n(t), \quad t \in [-h, 0], \tag{3.2}$$

assuming that P_n is generated by $\{e_k; k = 1, \ldots, n\}$ of H and P'_n its restriction on V – projectors of H and V correspondingly:

(1) $A^n = P'_n A$;

(2) $u^n(t) = P'_n u(t)$;

(3) $\phi^n(t) = P'_n \phi(t)$;

(4) $f^n = P_n f$;

(5) $g^n = P_n g$;

(6) $\sigma^n = P_n \sigma$.

Then prove that each of $(3.1), (3.2)$ has exactly one solution.

Step 2: Then we create a priory estimate on solutions of projected equations, which looks as follows:

$$E \sup_{t\in[0,t_1]} \big(\|u^n(t)\|_V^2 + \|g^n u_t^n\|_V^2\big) + E\bigg(\int_0^{t_1} \|u^n(t)\|_V^2 \, dt\bigg) \leq A$$

for some $A > 0$.

Those estimates are uniform (not dependant on dimension) and t_1 depends only on predefined coercitivity constants from (A3) and (C3), which implies that sequence of solutions are weak compact, hence holds weakly converging subsequence and can be iteratively continued on further intervals.

Step 3: After that we prove that we can make $n \to \infty$ in projected equations.

There we use the monotonicity condition and the growth conditions (G2) and (C2).

Additionally we prove that the energy equation holds, which implies existence and continuous dependence on initial data.

Corollary (Continuous dependence on the initial data)**.** *Let the conditions of the theorem above hold. Let ϕ and ϕ_1 be initial data for the solutions $u(t,\phi)$ and $u(t,\phi_1)$ of equation* (0.1)*,* (0.2)*. Then there exist a constant $C(T)$ such that*

$$E \sup_{t\in[0,T]} \big(\|u_t(\phi) - u_t(\phi_1)\|_C^2\big) \leq C(T)\|\phi - \phi_1\|_C^2.$$

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