The Method of Local Linear Approximation in the Theory of Nonlinear Impulse Systems

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Abstract

For nonlinear differential equations with impulsive perturbations, a general assertion about the existence of bounded solutions is given. With the help of this assertion necessary and sufficient conditions for the existence and uniqueness of bounded solutions of analogous linear equations are obtained. The equations are studied using the method of local linear approximation of nonlinear equations.

1 Problem statement

A method of studying nonlinear differential equations with impulse disturbances is proposed, which uses the approximation of these equations by linear systems on spheres with radii dependent on these systems. In the case of linear momentum equations, this method provides not only sufficient, but also necessary conditions for the existence and unity of bounded solutions of the corresponding equations.

2 Basic notation, spaces and problem

Let $\mathbb{R}$ and $\mathbb{Z}$ – the set of all real and integer numbers, respectively, $\mathbb{T} = \{t_n : \ n \in \mathbb{Z}\}$ – the set of real numbers for which $t_n < t_{n+1}$ for all $n \in \mathbb{Z}$, $\lim\limits_{n\to-\infty} t_n = -\infty$ and $\lim\limits_{n\to+\infty} t_n = +\infty$, E – a finite-dimensional Banach space over the field of real or complex numbers with norm $\|\cdot\|_E$ and $L(X,Y)$ – Banach space of linear continuous operators $A : X \to Y$ with the norm

$$\|A\|_{L(X,Y)} = \sup_{\|x\|_X=1} \|Ax\|_Y,$$

where X and Y – Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ in accordance.

Denote through $C^0(\mathbb{R},\mathbb{T},E)$ the Banach space of defined, continuous and bounded on $\mathbb{R}\setminus\mathbb{T}$ functions $x = x(t)$ with values in E, for each of which there are finite boundaries $\lim\limits_{t\to t_n-0} x(t) = x(t_n-0)$ and $\lim\limits_{t\to t_n+0} x(t) = x(t_n+0)$ to all $n \in \mathbb{Z}$, with the norm

$$\|x\|_{C^0(\mathbb{R},\mathbb{T},E)} = \sup_{t\in\mathbb{R}\setminus\mathbb{T}} \|x(t)\|_E,$$

through $C^1(\mathbb{R},\mathbb{T},E)$ denote the Banach space of continuously differentiable by $\mathbb{R}\setminus\mathbb{T}$ functions $x\in C^0(\mathbb{R},\mathbb{T},E)$, for each of which $dx/dt\in C^0(\mathbb{R},\mathbb{T},E)$, with the norm

$$\|x\|_{C^1(\mathbb{R},\mathbb{T},E)}=\max\Big\{\sup_{t\in\mathbb{R}\setminus\mathbb{T}}\|x(t)\|_E,\ \sup_{t\in\mathbb{R}\setminus\mathbb{T}}\Big\|\frac{dx(t)}{dt}\Big\|_E\Big\},$$

and through $\mathfrak{M}(\mathbb{Z},E)$ – Banach space of two-way sequences $\mathfrak{g}=g_n$ elements g_n, $n\in\mathbb{Z}$, space E with the norm

$$\|\mathfrak{g}\|_{\mathfrak{M}(\mathbb{Z},E)}=\sup_{n\in\mathbb{Z}}\|g_n\|_E.$$

Let us also consider the Banach space $C^i(\mathbb{R},\mathbb{T},E)\times\mathfrak{M}(\mathbb{Z},E)$, where $i\in\{0,1\}$, pairs $(x,\mathfrak{g})$ of elements $x=x(t)\in C^i(\mathbb{R},\mathbb{T},E)$ and $\mathfrak{g}=g_n\in\mathfrak{M}(\mathbb{Z},E)$ with the norm

$$\|(x,\mathfrak{g})\|_{C^i(\mathbb{R},\mathbb{T},E)\times\mathfrak{M}(\mathbb{Z},E)}=\max\big\{\|x\|_{C^i(\mathbb{R},\mathbb{T},E)},\|\mathfrak{g}\|_{\mathfrak{M}(\mathbb{Z},E)}\big\}.$$

For function jumps $x\in C^0(\mathbb{R},\mathbb{T},E)$ in the points of the set $\mathbb{T}$ similarly, as in [2,3], we will use the notation

$$\Delta x|_{t=t_n}=x(t_n+0)-x(t_n-0),\ \ n\in\mathbb{Z}.$$

Consider a continuous display $F:(\mathbb{R}\setminus\mathbb{T})\times E\to E$, for which for every bounded set $\mathcal{M}\subset E$ a function $F(t,x)$ is bounded on the set $(\mathbb{R}\setminus\mathbb{T})\times\mathcal{M}$ and this function is uniformly continuous on every bounded subset $\mathcal{N}$ plural $(\mathbb{R}\setminus\mathbb{T})\times E$. Also consider continuous mappings $G_n:E\to E$, $n\in\mathbb{Z}$, for which $\sup\limits_{n\in\mathbb{Z},\ x\in\mathcal{M}}\|G_n(x)\|_E<+\infty$ for every bounded set $\mathcal{M}\subset E$.

From the conditions that satisfy F, it follows that for each $x\in C^0(\mathbb{R},\mathbb{T},E)$ the function $y=F(t,x(t))$ is an element of the space $C^0(\mathbb{R},\mathbb{T},E)$.

We will be interested in the conditions under which the system of differential equations with an impulse disturbance is fulfilled

$$\begin{cases}\dfrac{dx(t)}{dt}+F(t,x(t))=f(t), & t\in\mathbb{R}\setminus\mathbb{T},\\ \Delta x|_{t=t_n}+G_n(x(t_n-0))=g_n, & n\in\mathbb{Z}\end{cases}\tag{2.1}$$

for each function $f=f(t)\in C^0(\mathbb{R},\mathbb{T},E)$ and sequences $\mathfrak{g}=g_n\in\mathfrak{M}(\mathbb{Z},E)$ will have at least one solution $x=x(t)\in C^1(\mathbb{R},\mathbb{T},E)$.

The left part of the system of equations (2.1) operator is generated $\mathscr{I}$, that works with $C^1(\mathbb{R},\mathbb{T},E)$ in $C^0(\mathbb{R},\mathbb{T},E)\times\mathfrak{M}(\mathbb{Z},E)$. If you use operators $\mathcal{L}:C^1(\mathbb{R},\mathbb{T},E)\to C^0(\mathbb{R},\mathbb{T},E)$ and $\mathcal{D}:C^0(\mathbb{R},\mathbb{T},E)\to\mathfrak{M}(\mathbb{Z},E)$, which are defined by equalities

$$(\mathcal{L}x)(t)=\frac{dx(t)}{dt}+F(t,x(t)),\ \ t\in\mathbb{R}\setminus\mathbb{T},$$

and

$$(\mathcal{D}x)_n=\Delta x\big|_{t=t_n}+G_n(x(t_n-0)),\ \ n\in\mathbb{Z},$$

then according to (2.1) operator $\mathscr{I}:C^1(\mathbb{R},\mathbb{T},E)\to C^0(\mathbb{R},\mathbb{T},E)\times\mathfrak{M}(\mathbb{Z},E)$ is given by the ratio

$$\mathscr{I}x=(\mathcal{L}x,\mathcal{D}x),\ \ x\in C^1(\mathbb{R},\mathbb{T},E).$$

Let $R(\mathscr{I})$ – set of operator values $\mathscr{I}$, i.e. $\{\mathscr{I}x:x\in C^1(\mathbb{R},\mathbb{T},E)\}$.

System of equations (2.1) and the corresponding operator $\mathscr{I}$ in the general case are nonlinear and clarification for system (2.1) conditions for the existence of bounded solutions for each function $f=f(t)\in C^0(\mathbb{R},\mathbb{T},E)$ and sequences$\mathfrak{g}=g_n\in\mathfrak{M}(\mathbb{Z},E)$ or similarly, finding out the conditions of execution for the operator$\mathscr{I}$ equality

$$R(\mathscr{I})=C^0(\mathbb{R},\mathbb{T},E)\times\mathfrak{M}(\mathbb{Z},E)$$

are not trivial tasks.

3 The main result

When finding out the conditions for the existence of limited solutions of system (2.1) we will use the auxiliary linear systems with impulse disturbance of appearance

$$\begin{cases} \dfrac{dx(t)}{dt} + A(t)x(t) = f(t), & t \in \mathbb{R} \setminus \mathbb{T}, \\ \Delta x\big|_{t=t_n} + B_n x(t_n - 0) = g_n, & n \in \mathbb{Z}, \end{cases} \tag{3.1}$$

coefficients $A(t)$ and B_n of which in a certain sense (see the formulation of Theorem 3.1 and the relation (3.5)) differ little on closed spheres of space E from $F(t,\cdot)$ and $G_n(\,\cdot\,)$ in accordance.

Let's use a set of pairs $(A,\mathfrak{B})$ defined and continuous on $\mathbb{R}\setminus\mathbb{T}$ functions $A = A(t)$ with values in $L(E,E)$ and bilateral sequences $\mathfrak{B} = B_n \in L(E,E)$, $n \in \mathbb{Z}$, which are elements of spaces $C^0(\mathbb{R},\mathbb{T},L(E,E))$ and $\mathfrak{M}(\mathbb{Z},L(E,E))$ in accordance.

For a pair of $(A,\mathfrak{B})$ let's match the linear continuous operator

$$\mathfrak{L}_{(A,\mathfrak{B})} : C^1(\mathbb{R},\mathbb{T},E) \to C^0(\mathbb{R},\mathbb{T},E) \times \mathfrak{M}(\mathbb{Z},E),$$

which is given by the ratio

$$\mathfrak{L}_{(A,\mathfrak{B})}x = (\mathscr{L}x, \mathscr{D}x), \ \ x \in C^1(\mathbb{R},\mathbb{T},E), \tag{3.2}$$

where

$$(\mathscr{L}x)(t) = \frac{dx(t)}{dt} + A(t)x(t), \ \ t \in \mathbb{R}\setminus\mathbb{T}, \tag{3.3}$$

and

$$(\mathscr{D}x)_n = \Delta x|_{t=t_n} + B_n x(t_n - 0), \ \ n \in \mathbb{Z}. \tag{3.4}$$

Set of linear operators $\mathfrak{L}_{(A,\mathfrak{B})} : C^1(\mathbb{R},\mathbb{T},E) \to C^0(\mathbb{R},\mathbb{T},E) \times \mathfrak{M}(\mathbb{Z},E)$, dependent on $(A,\mathfrak{B})$, each of which is determined by the left part of system (3.1), i.e. ratios (3.2)–(3.4), and has an inverse continuous operator $\mathfrak{L}^{-1}_{(A,\mathfrak{B})} : C^0(\mathbb{R},\mathbb{T},E) \times \mathfrak{M}(\mathbb{Z},E) \to C^1(\mathbb{R},\mathbb{T},E)$, denote by $\mathcal{O}$.

Theorem 3.1 ([1])**.** *Suppose for each number $H > 0$ there are such number $r > 0$ and $\mathfrak{L}_{(A,\mathfrak{B})} \in \mathcal{O}$ that*

$$\sup_{x\in\mathcal{B}^0[0,r]} \max\Big\{ \sup_{t\in\mathbb{R}\setminus\mathbb{T}} \|F(t,x(t)) - A(t)x(t)\|_E, \sup_{n\in\mathbb{Z}} \|G_n(x(t_n-0)) - B_n x(t_n-0)\|_E \Big\} \\ \leq r \|\mathfrak{L}^{-1}_{(A,\mathfrak{B})}\|^{-1}_{L(C^0(\mathbb{R},\mathbb{T},E)\times\mathfrak{M}(\mathbb{Z},E),C^1(\mathbb{R},\mathbb{T},E))} - H. \tag{3.5}$$

Then for each $f \in C^0(\mathbb{R},\mathbb{T},E)$ and $\mathfrak{g} \in \mathfrak{M}(\mathbb{Z},E)$ the system of equations (2.1) *has at least one solution $x \in C^1(\mathbb{R},\mathbb{T},E)$.*

Remark 3.1. In system (2.1) the reflection $F(t,\cdot)$, $t \in \mathbb{R}\setminus\mathbb{T}$, and $G_n(\,\cdot\,)$, $n \in \mathbb{Z}$, may be non-Lipschitz.

4 The case of linear impulse systems

Let's fix an arbitrary function $Q = Q(t) \in C^0(\mathbb{R},\mathbb{T},L(E,E))$ and a sequence

$$\mathfrak{R} = R_n \in \mathfrak{M}(\mathbb{Z},L(E,E)).$$

Consider the corresponding system of linear differential equations with an impulse disturbance

$$\begin{cases} \dfrac{dx(t)}{dt} + Q(t)x(t) = f(t), & t \in \mathbb{R} \setminus \mathbb{T}, \\ \Delta x\big|_{t=t_n} + R_n x(t_n - 0) = g_n, & n \in \mathbb{Z}, \end{cases}$$

where $f = f(t) \in C^0(\mathbb{R}, \mathbb{T}, E)$ and $\mathfrak{g} = g_n \in \mathfrak{M}(\mathbb{Z}, E)$, and the linear differential operator $\mathfrak{L}_{(Q,\mathfrak{R})} : C^1(\mathbb{R}, \mathbb{T}, E) \to C^0(\mathbb{R}, \mathbb{T}, E) \times \mathfrak{M}(\mathbb{Z}, E)$, which is given by the ratio

$$\mathfrak{L}_{(Q,\mathfrak{R})} x = (\mathscr{L}_1 x, \mathscr{D}_1 x), \;\; x \in C^1(\mathbb{R}, \mathbb{T}, E),$$

where

$$(\mathscr{L}_1 x)(t) = \frac{dx(t)}{dt} + Q(t)x(t), \;\; t \in \mathbb{R} \setminus \mathbb{T},$$

and

$$(\mathscr{D}_1 x)_n = \Delta x\big|_{t=t_n} + R_n x(t_n - 0), \;\; n \in \mathbb{Z}.$$

Let's use Theorem 3.1 and operators $\mathfrak{L}_{(A,\mathfrak{B})} \in \mathcal{O}$, which are determined by ratios (3.2)–(3.4). The following two statements are true.

Theorem 4.1 ([1])**.** *For each number $H > 0$ there are such number $r > 0$ and the operator $\mathfrak{L}_{(A,\mathfrak{B})} \in \mathcal{O}$, for which*

$$\sup_{x \in \mathcal{B}^0[0,r]} \max\Big\{ \sup_{t \in \mathbb{R}\setminus\mathbb{T}} \|Q(t)x(t)) - A(t)x(t)\|_E, \sup_{n \in \mathbb{Z}} \|R_n x(t_n - 0) - B_n x(t_n - 0)\|_E \Big\} < r \|\mathfrak{L}^{-1}_{(A,\mathfrak{B})}\|^{-1}_{L(C^0(\mathbb{R},\mathbb{T},E)\times\mathfrak{M}(\mathbb{Z},E), C^1(\mathbb{R},\mathbb{T},E))} - H,$$

if and only if the linear operator $\mathfrak{L}_{(Q,\mathfrak{R})} : C^1(\mathbb{R}, \mathbb{T}, E) \to C^0(\mathbb{R}, \mathbb{T}, E) \times \mathfrak{M}(\mathbb{Z}, E)$ has an inverse continuous operator.

Theorem 4.2 ([1])**.** *Operator $\mathfrak{L}_{(Q,\mathfrak{R})} : C^1(\mathbb{R}, \mathbb{T}, E) \to C^0(\mathbb{R}, \mathbb{T}, E) \times \mathfrak{M}(\mathbb{Z}, E)$ has an inverse continuous operator if and only if the operator exists $\mathfrak{L}_{(A,\mathfrak{B})} \in \mathcal{O}$, for which*

$$\sup_{x \in \mathcal{B}^0[0,1]} \max\Big\{ \sup_{t \in \mathbb{R}\setminus\mathbb{T}} \|Q(t)x(t)) - A(t)x(t)\|_E, \sup_{n \in \mathbb{Z}} \|R_n x(t_n - 0) - B_n x(t_n - 0)\|_E \Big\} < \|\mathfrak{L}^{-1}_{(A,\mathfrak{B})}\|^{-1}_{L(C^0(\mathbb{R},\mathbb{T},E)\times\mathfrak{M}(\mathbb{Z},E), C^1(\mathbb{R},\mathbb{T},E))}.$$

5 Perturbations of linear impulse systems are small at infinity

Consider a system of differential equations with an impulse disturbance

$$\begin{cases} \dfrac{dx(t)}{dt} + A(t)x(t) = F(t, x(t)) + f(t), & t \in \mathbb{R} \setminus \mathbb{T}, \\ \Delta x\big|_{t=t_n} + B_n(x(t_n - 0)) = G_n(x(t_n - 0)) + g_n, & n \in \mathbb{Z}, \end{cases} \tag{5.1}$$

in which function $A = A(t)$, $f = f(t)$ and sequences $\mathfrak{B} = B_n$, $\mathfrak{g} = g_n$, $n \in \mathbb{Z}$ are such as in system (3.1), and non-linear mappings $F(t, \cdot) : E \to E$, $t \in \mathbb{R} \setminus \mathbb{T}$, and $G_n(\cdot) : E \to E$, $n \in \mathbb{Z}$ are such as in system (2.1).

We assume that the operator $\mathfrak{L}_{(A,\mathfrak{B})} : C^1(\mathbb{R},\mathbb{T},E) \to C^0(\mathbb{R},\mathbb{T},E) \times \mathfrak{M}(\mathbb{Z},E)$, which is known as the left part of system (3.1), has an inverse continuous operator $\mathfrak{L}^{-1}_{(A,\mathfrak{B})}$ and

$$\varliminf_{r\to+\infty} r^{-1} \sup_{x\in\mathcal{B}^0[0,r]} \max\Big\{ \sup_{t\in\mathbb{R}\setminus\mathbb{T}} \|F(t,x(t))\|_E, \sup_{n\in\mathbb{Z}} \|G_n(x(t_n-0)\|_E \Big\} < \|\mathfrak{L}^{-1}_{(A,\mathfrak{B})}\|^{-1}_{L(C^0(\mathbb{R},\mathbb{T},E)\times\mathfrak{M}(\mathbb{Z},E),C^1(\mathbb{R},\mathbb{T},E))}. \tag{5.2}$$

A special case of Theorem 4.2 is

Theorem 5.1 ([1])**.** *System of equations* (5.1) *for each* $(f,\mathfrak{g}) \in C^0(\mathbb{R},\mathbb{T},E) \times \mathfrak{M}(\mathbb{Z},E)$ *has at least one solution* $x \in C^1(\mathbb{R},\mathbb{T},E)$.

Remark 5.1. Ratio (5.2) is performed if

$$\sup_{(t,x)\in\mathbb{R}\times E} \|F(t,x)\|_E + \sup_{(n,x)\in\mathbb{Z}\times E} \|G_n(x)\|_E < +\infty.$$

Remark 5.2. Reflection $F(t,\,\cdot\,) : E \to E$, $t \in \mathbb{R}\setminus\mathbb{T}$, and $G_n(\,\cdot\,) : E \to E$, $n \in \mathbb{Z}$, in system (5.1) can be such that the relation (5.2) holds and

$$\varlimsup_{r\to+\infty} r^{-1} \sup_{x\in\mathcal{B}^0[0,r]} \max\Big\{ \sup_{t\in\mathbb{R}\setminus\mathbb{T}} \|F(t,x(t))\|_E, \sup_{n\in\mathbb{Z}} \|G_n(x(t_n-0))\|_E \Big\} = +\infty.$$

Remark 5.3. The method of local linear approximation in the theory of nonlinear differential, difference, and differential functional equations is considered in [4].

Theorems 3.1, 4.1, 4.2, 5.1 are substantiated using the theory of c-continuous operators, the elements of which are laid out in [1, 4].

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