Investigation of the Behavior of Solutions of Stochastic Ito Differential Equations

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1 Introduction

In this paper, we investigate the conditions of the existence and the general appearance of locally invariant curves of a perturbed differential equation by a random Wiener process of the "white noise" type in the form of Ito. Random perturbations occur along the phase velocity vector of the corresponding undisturbed differential (deterministic) equation. In [3], the conditions for the existence and uniqueness of solutions of stochastic differential equations are presented. Construction and study of the phase portrait of stochastic Ito differential equations with a degenerate diffusion matrix was carried out in [4]. For nonlinear stochastic Ito differential equations with Markov switching, some sufficient conditions for invariance, stochastic stability, stochastic asymptotic stability, and instability of invariant sets of equations are obtained in [5]. There is the significant literature devoted to the invariant sets of ordinary differential equations, functional differential equations, and stochastic differential equations, and we here mention [2, 5, 7]. The conditions for the existence of bounded solutions of linear and nonlinear pulsed systems were obtained in [1, 6].

In this paper, the conditions under which the locally phase trajectories of the corresponding deterministic differential equation can be locally invariant curves of the perturbed equation are established. A model example describing a certain class of problems related to the study of random harmonic oscillators is given. The conducted researches in an example illustrate application of the received results for construction and the analysis of stochastic differential equations of Ito.The obtained conditions make it possible to build classes of stochastic differential equations for which the given set is invariant.

2 Setting of the problem and the main results

Consider a system of stochastic differential equations

$$d\xi(t) = a(\xi(t)\,dt + b(\xi(t)\,dw(t),\ \ \xi(0) = x^0, \tag{2.1}$$

where $a(x) = (a_1(x), a_2(x))$, $b(x) = (b_1(x), b_2(x))$ – continuous-differential functions in a certain open domain $D \subset R_2$. Denote by $w(t)$ the one-dimensional Wiener process defined in probabilistic space (Ω, F, P), $x = (x_1, x_2)$ – point in D, $x^0 \in D$. It is known [3] that under the given conditions for coefficients of the equation, there is a continuous with probability 1 unique strong solution $\xi(t)$ for all $t \geq 0$ of this equation.

Denote by $\Gamma_D(G)$ the set of the form $\Gamma = \{x : G(x) = C\} \subset D$, where C is a definite constant, $G(x)$ – a twice continuous-differential function in D and has no special points for all $x \in \Gamma$.

If for all $x^0 \in \Gamma_D(G)$

$$P\Big\{\sup_{0\leq t\leq \tau_{D(x^0)}} \big|G(\xi(t)) - G(x^0)\big| = 0\Big\} = 1,$$

where $\tau_{D(x^0)}$ is the moment of the first exit of the solution from the domain D, then the curve $\Gamma_{D(G)}$ is a locally invariant curve of the corresponding equation (2.1).

Consider the problem of investigating the conditions under which the locally phase trajectories of a deterministic differential equation can be locally invariant curves of the corresponding perturbed equation by a random Wiener process of the "white noise" type in the Ito form.

According to [4], the locally invariant curve $\Gamma_{D(G)}$ of equation (2.1) coincides with the locally phase trajectory of equation

$$\frac{dx(t}{dt} = b(x(t)), \quad x(0) = x^0. \tag{2.2}$$

That is $G(x(t)) = G(x^0)$, for all for $t \geq 0$ whom $x(t) \in D$. Since, $(\nabla G(x), b(x)) = 0$, then the phase velocity vector $b(x)$ of equation (2.2) is directed along the tangent to the phase trajectory $G(x) = G(x^0)$ at point x. Thus we obtained the following theorem.

Theorem 2.1. *Locally phase trajectory $\Gamma_{D(G)}$ of equation (2.2), in which $|b(x)| > 0$ for all $x \in \Gamma_{D(G)}$, there can be a local phase curve of equation (2.1), only when the random perturbation of equation (2.2) by Ito-shaped "white noise" processes occurs along the phase velocity vector of equation (2.2).*

We obtain the following result for the case $(\nabla G(x), a(x)) = 0$ for all $x \in \Gamma_{D(G)}$.

Since we have a given function $G(x)$, it follows from the necessary condition [4] that

$$b(x) = \big(- G'_{x_2}(x)g(x), G'_{x_1}(x)g(x)\big)$$

for each $x \in \Gamma_D(G)$, where $g(x)$ is an arbitrary continuous-differential function.

Therefore, from the necessary condition of local invariance [4], we have equality $Q(x)g^2(x) = 0$ for all $x \in \Gamma_{D(G)}$, where

$$Q(x) = G''_{x_1x_1}(x)(G'_{x_2})^2(x) + G''_{x_2x_2}(x)(G'_{x_1})^2(x) - 2G''_{x_1x_2}(x)G'_{x_1}(x)G'_{x_2}(x).$$

Theorem 2.2. *The locally phase trajectory $\Gamma_D(G)$ of equation (2.2) can be a locally invariant curve of equation (2.1) in which $(\nabla G(x), a(x)) = 0$ for all $x \in \Gamma_D(G)$, only when the curve consists only of equilibrium points of equation (2.2) ($|b(x)| = 0$), and points where the curvature of the curve $\Gamma_D(G)$ is zero.*

Theorem 2.3. *Let the curves $\Gamma_D(G)$ be the set of locally phase trajectories of equation (2.2) for all C. If the curvature of the curve $\Gamma_D(G)$ is not equal to zero at the point $x^0 \in D$, $|b(x^0)| > 0$ and $(\nabla G(x), a(x)) = 0$ for all $x \in D$, then the solution of equation (2.2) instantly deviates from $\Gamma_D(G)$ the direction of convexity of the curve at the point x^0.*

In order for the solution of equation (2.2) to remain on the phase trajectory $\Gamma_D(G)$ in case of random perturbations along the phase velocity vector $b(x)$ by the Ito-shaped "white noise" process, it is necessary to additionally introduce the corresponding control vector $a(x)$ in equation (2.2).

3 Application to the perturbed limit cycle

For qualitative analysis of stochastic differential equations, it is convenient to use the polar coordinate system $x_1 = r\cos\phi$, $x_2 = r\cos\phi$.

Therefore, we present an auxiliary statement about the connection of the stochastic differential equation (2.1) with the corresponding stochastic differential equation in polar coordinates. We consider a system of stochastic differential equations in the domain $D = \{r > 0, -\infty < \phi < +\infty\}$:

$$\begin{cases} dr(t) = a_1(r,\phi)\,dt + b_1(r,\phi)\,dw(t) \\ d\phi(t) = a_2(r,\phi)\,dt + b_2(r,\phi)\,dw(t), \end{cases} \tag{3.1}$$

where the flow σ-algebra F_t and the one-dimensional process $w(t)$ are the same as in equation (2.1). The coefficients of system are such that there is one strong solution of the system until the moment of the first exit τ_D from the domain D.

Process $\xi(t) = (r(t)\cos\phi(t), r(t)\sin\phi(t))$ for $t < \tau_D$ is the solution of the stochastic equation (2.1).

If for $t < \tau_D$ there is a unique solution of equation (2.1), then for $t < \tau_D$, $r(t)$ is the radial characteristic of the process $\xi(t)$ and $\phi(t)$ is the angular characteristic of the process $\xi(t)$.

Consider equation (2.1) with the corresponding coefficients:

$$\begin{gathered}a_1(x) = x_2 q(x) + \alpha x_1(1 - |x|^2), \quad a_2(x) = -x_1 q(x) + \alpha x_2(1 - |x|^2),\\ b_1 = x_2 g, \quad b_2 = -x_1 g,\end{gathered}$$

where α, g are constants, $q(x)$ is arbitrary continuous-differential function in R^2 and $|x| = \sqrt{x_1^2 + x_2^2}$.

The given system describes a certain class of harmonic oscillators that depend on the parameters α, g.

In this case, the phase trajectories of the corresponding deterministic equation (2.2) are circles $x_1^2 + x_2^2 = C$, where $C > 0$ and equilibrium point $(0;0)$.

To study the phase "picture" of this equation (2.1), consider the process$\eta(t) = G(\xi(t))$, where $G(x) = x_1^2 + x_2^2$.

According to the formula Ito we obtain the equation:

$$d\eta(t) = \eta(t)\big[2\alpha(1 - \eta(t)) + g^2\big]\,dt, \quad \eta(0) = |x^0|^2. \tag{3.2}$$

The invariant set of equation (2.1) is the circle $|x|^2 = 1 + g^2(2\alpha)^{-1}$ at $\alpha > 0$ and at $2\alpha < -g^2$. If $2\alpha = -g^2$ or $\alpha = 0$, then the invariant set will be a point $(0;0)$.

If $-g^2 < 2\alpha < 0$, then there are no invariant curves for this equation (2.1).

Suppose $\alpha = 0$, then from equation (3.2) we have

$$\eta(t) = |x^0|^2 e^{gt^2}$$

for all $t \geq 0$ and therefore $\eta(t) \to \infty$ for $t \to \infty$.

If $\alpha \neq 0$ and $|x^0| > 0$, then with probability 1 for all $t \geq 0$ it holds

$$\eta(t) = \frac{1 + g^2(2\alpha)^{-1}}{1 + C_0\exp\{-(2\alpha + g^2)t\}}\,, \tag{3.3}$$

where

$$C_0 = |x^0|^{-2}\big[1 + g^2(2\alpha)^{-1} - |x^0|^2\big].$$

From the analysis of solution (3.3), we have the following:

(a) If $1 + g^2(2\alpha)^{-1} > 0$, then $|x^0|^2 = 1 + g^2(2\alpha)^{-1}$ is an invariant circle and $|\xi(t)|^2 = 1 + g^2(2\alpha)^{-1}$ with a probability of 1 for all $t \geq 0$.

If in this case $\alpha > 0$ and $|x^0|^2 \neq 1 + g^2(2\alpha)^{-1}$, then $|\xi(t)|^2 \to 1 + g^2(2\alpha)^{-1}$ with a probability of 1 at $t \to \infty$ (stability with probability 1).

If $\alpha < 0$ and $|x^0|^2 < 1 + g^2(2\alpha)^{-1}$, then $|\xi(t)|^2 \to 0$ with a probability of 1 at $t \to \infty$.

If $\alpha < 0$ and $|x^0|^2 > 1 + g^2(2\alpha)^{-1}$, then $|\xi(t)|^2 \to \infty$ with a probability of 1 at $t \to t_0$, where

$$t_0 = \frac{-\ln(-1/C_0)}{2 + g^2}\,.$$

(b) If $1 + g^2(2\alpha)^{-1} < 0$ and $\alpha < 0$, then there are no invariant curves.

The equation for the process argument $\xi(t) = (r(t)\cos\phi(t), r(t)\sin\phi(t))$ in this case takes the form

$$d\phi(t) = -q_1(\eta(t), \phi(t))\,dt - g\,dw(t), \tag{3.4}$$

where

$$q_1(\eta(t), \phi(t)) = q\Big(\sqrt{\eta(t)\cos\phi(t)}, \sqrt{\eta(t)\sin\phi(t)}\Big).$$

The systems of equations (3.2), (3.4) provide opportunities for a more detailed study of the behavior of the solution $\xi(t)$.

In particular, if $q(x) = q_0$, where q_0 - constant, then $\frac{w(t)}{t} \to 0$ with probability of 1 at $t \to \infty$ and $\frac{\phi(t)}{t} \to -q_0$ with probability of 1 at $t \to \infty$.

In the case of $q(x) = 0$, process $\phi(t)$ has a normal distribution$N(\phi(0), g^2 t)$ for all $t > 0$.

Note that when $|x^0| = 1 + g^2(2\alpha)^{-1}$ we obtain $\eta(t) = (2\alpha)^{-1}g^2$ with probability of 1 for all $t > 0$.

Equation (3.4) will turn into an equation with one variable $\phi(t)$, which greatly simplifies its study.

By changing the values of the parameters of this example, we can obtain various models of stochastic oscillators.

References

[1] F. Asrorov, O. Perehuda, V. Sobchuk and A. Sukretna, Establishing conditions for the existence of bounded solutions to the weakly nonlinear pulse systems. *Eastern-European Journal of Enterprise Technologies* **4** (2021), no. 4(112), 6–12.

[2] P. W. Bates, K. Lu and Ch. Zeng, Existence and persistence of invariant manifolds for semiflows in Banach space. *Mem. Amer. Math. Soc.* **135** (1998), no. 645, 1–129.

[3] I. I. Gihman and A. V. Skorokhod, *Stochastic Differential Equations and their Applications.* Naukova Dumka, Kyiv, 1982.

[4] G. L. Kulinich and O. V. Pereguda, Phase picture of the diffusion processes with the degenerate diffusion matrices. *Random Oper. Stochastic Equations* **5** (1997), no. 3, 203–216.

[5] J. Luo, Stability of invariant sets of Itô stochastic differential equations with Markovian switching. *J. Appl. Math. Stoch. Anal.* **2006**, Art. ID 59032, 6 pp.

[6] A. M. Samoĭlenko, V. G. Samoĭlenko and V. V. Sobchuk, On periodic solutions of an equation of a nonlinear oscillator with impulse action. (Ukrainian) *Ukraïn. Mat. Zh.* **51** (1999), no. 6, 827–834; translation in *Ukrainian Math. J.* **51** (1999), no. 6, 926–933 (2000).

[7] O. M. Stanzhits'kiĭ, Investigation of invariant sets of Itô stochastic systems by means of Lyapunov functions. (Ukrainian) *Ukraïn. Mat. Zh.* **53** (2001), no. 2, 282–285; translation in *Ukrainian Math. J.* **53** (2001), no. 2, 323–327.