Rapidly Growing Solutions to Two-Dimensional Nonlinear Differential Systems

Nino Partsvania[1,2]

[1]*Andrea Razmadze Mathematical Institute of Ivane Javakhishvili Tbilisi State University Tbilisi, Georgia*

[2]*International Black Sea University, Tbilisi, Georgia*

E-mail: nino.partsvania@tsu.ge

Let $a > 0$, and let $f_i : [a,+\infty[\times]0,+\infty[\to]0,+\infty[\ \ (i = 1,2)$ be continuous functions satisfying the local Lipschitz condition in the second argument.

We consider the differential system

$$u_1' = f_1(t,u_2), \ \ u_2' = f_2(t,u_1). \tag{1}$$

A solution to that system in an arbitrary interval $I \subset [a,+\infty[$ is sought on the set of two-dimensional continuously differentiable vector functions with positive components.

A solution (u_1,u_2) to system (1) defined on some infinite interval $[t_0,+\infty[\subset[a,+\infty[$ is said to be **proper**. Obviously, the components of an arbitrary proper solution (u_1,u_2) to system (1) are increasing functions and satisfy one of the following two conditions:

$$\lim_{t\to+\infty} u_i(t) = +\infty \ \ (i = 1,2);$$

$$\lim_{t\to+\infty} u_k(t) < +\infty \ \text{ for some } \ k \in \{1,2\}.$$

In the first case the above mentioned solution is said to be **rapidly growing**, while in the second case it is said to be **slowly growing**.

A solution (u_1,u_2) to system (1) defined on some finite interval $[t_0,t_1[\subset[a,+\infty[$ is said to be **blow-up** if

$$\lim_{t\to t_1} \big(u_1(t) + u_2(t)\big) = +\infty.$$

By a solution to the system under consideration we mean a solution that is maximally extended to the right. Thus every solution to that system is either proper or blow-up.

A particular case of system (1) is the second order differential equation

$$u'' = f(t,u) \tag{2}$$

with a continuous right-hand side $f : [a,+\infty[\times]0,+\infty[\to]0,+\infty[$.

A solution to that equation in an arbitrary interval $I \subset [a,+\infty[$ is sought on the set of twice continuously differentiable functions, satisfying the inequalities

$$u(t) > 0, \ \ u'(t) \geq 0,$$

and by a solution it is meant a maximally extended to the right solution.

According to the above definitions, a solution to equation (2) defined on some infinite interval $[t_0,+\infty[\subset[a,+\infty[$ is said to be **proper**. A proper solution u to equation (2) is said to be **rapidly growing** if

$$\lim_{t\to+\infty} u'(t) = +\infty,$$

and it is said to be **slowly growing** otherwise. As for the solution to equation (2) defined on some finite interval $[t_0,t_1[\subset [a,+\infty[$, it is said to be **blow-up** if

$$\lim_{t\to t_1} u(t)=+\infty.$$

R. Emden and R. H. Fowler have investigated in detail asymptotic properties of proper monotone solutions to the frequently occurring in applications differential equation

$$u''=t^\sigma u^\lambda.$$

The results obtained by them are reflected in the monograph by R. Bellman ([2], Ch. VII). The theory of monotone solutions to the Emden–Fowler type differential equation with general coefficient

$$u''=p(t)u^\lambda$$

was constructed by I. T. Kiguradze [8] (see, also [13], Ch. V). The asymptotic theory of nonoscillatory and oscillatory solutions to two-dimensional differential systems was constructed by J. D. Mirzov [15].

The foundations of the asymptotic theory of monotone solutions to an arbitrary order differential equations were laid back in the late sixties of the last century and it still remains relevant (see [1, 3–7, 9–14] and the references therein).

The results on the existence of rapidly growing solutions and on their asymptotic estimates given in the present work are obtained based on the method proposed by I. T. Kiguradze and G. G. Kvinikadze [14].

We investigate the case, where

$$f_1(t,x)\ge f_1(s,y) \ \text{ for } \ t\ge s, \ \ x\ge y, \quad f_2(t,x)\ge f_2(s,y) \ \text{ for } \ t\le s, \ \ x\ge y. \tag{3}$$

Consequently, the function f_1 is assumed to be nondecreasing in both arguments, while the function f_2 is assumed to be nonincreasing in first argument and nondecreasing in the second argument.

Everywhere below we use the following notation.

$$f_{0i}(t,x)=\int\limits_0^x f_i(t,y)\,dy \ \text{ for } \ t\ge a, \ \ x>y;$$

φ_0 is a function defined from the equality

$$f_{01}(t,\varphi_0(t,x))=x \ \text{ for } \ t\ge a, \ \ x>0;$$
$$\varphi(t,x)=f_1\big(t,\varphi_0(t,f_{02}(t,x))\big) \ \ t\ge a, \ \ x>0.$$

Theorem 1. *Let conditions* (3) *be fulfilled and let the differential equation*

$$v'=\varphi(t,v) \tag{4}$$

have no proper solution. Then any solution to the differential system (1) *is blow-up.*

Theorem 2. *Let conditions* (3) *be fulfilled and let the differential equation* (4) *have a unique solution, satisfying the limit condition*

$$\lim_{t\to+\infty} v(t)=+\infty. \tag{5}$$

Then for any $t_0 \in [a,+\infty[$ there exists a positive number γ such that if

$$c_1 \geq 0, \quad c_2 \geq \gamma, \tag{6}$$

then the solution (u_1, u_2) to the differential system (1), satisfying the initial conditions

$$u_1(t_0) = c_1, \quad u_2(t_0) = c_2, \tag{7}$$

is blow-up.

Theorem 3. *Let along with (3) the condition*

$$\int_a^{+\infty} f_2\Big(t, x + \int_a^t f_1(s,x)\,ds\Big)\,dt < +\infty \quad \text{for } x > 0$$

hold. If, moreover, problem (4), (5) has a unique solution v, then the differential system (1) along with two-parametric set of slowly growing solutions has a one-parametric set of rapidly growing solutions whose first component for large t_0 admits the estimate

$$u_1(t) \leq v(t) \quad \text{for } t \geq t_0.$$

As an example, we consider the Emden–Fowler type differential system

$$u_1' = p_1(t) u_2^{\lambda_1}, \quad u_2' = p_2(t) u_1^{\lambda_2}, \tag{8}$$

where λ_1 and λ_2 are positive numbers such that

$$\lambda_1 \lambda_2 > 1,$$

$p_1 : [a,+\infty[\to]0,+\infty[$ is a nondecreasing continuous function, and $p_2 : [a,+\infty[\to]0,+\infty[$ is a nonincreasing continuous function.

System (8) can be obtained from system (1) in the case, where

$$f_1(t,x) = p_1(t) x^{\lambda_1}, \quad f_2(t,x) = p_2(t) x^{\lambda_2}.$$

In that case the above defined functions f_{0i} $(i = 1,2)$, φ_0, φ have the form

$$\begin{aligned}
f_{0i}(t,x) &= \frac{1}{1+\lambda_1} p_i(t) x^{1+\lambda_i} \quad (i=1,2),\\
\varphi_0(t,x) &= \Big(\frac{1+\lambda_1}{p_1(t)}\Big)^{\frac{1}{1+\lambda_1}} x^{\frac{1}{1+\lambda_1}},\\
\varphi(t,x) &= \Big(\frac{1+\lambda_1}{1+\lambda_2}\Big)^{\frac{\lambda_1}{1+\lambda_2}} \big(p_1(t) p_2^{\lambda_1}(t)\big)^{\frac{1}{1+\lambda_1}} x^{\frac{\lambda_1+\lambda_1\lambda_2}{1+\lambda_1}}.
\end{aligned}$$

Thus Theorems 1–3 yield the following statements.

Corollary 1. *If*

$$\int_a^{+\infty} \big(p_1(t) p_2^{\lambda_1}(t)\big)^{\frac{1}{1+\lambda_1}}\,dt = +\infty,$$

then any solution to system (8) is blow-up.

Corollary 2. *If*

$$\int\limits_a^{+\infty} \big(p_1(t)p_2^{\lambda_1}(t)\big)^{\frac{1}{1+\lambda_1}}\,dt < +\infty, \tag{9}$$

then for any $t_0 \in [a, +\infty[$ there exists a positive number γ such that if inequalities (6) *are satisfied, then the solution to problem* (8), (7) *is blow-up.*

Corollary 3. *Let along with* (9) *the condition*

$$\int\limits_a^{+\infty} p_2(t)\bigg(\int\limits_a^t p_1(s)\,ds\bigg)^{\lambda_2}\,dt < +\infty$$

hold. Then the differential system (8) *along with two-parametric set of slowly growing solutions has a one-parametric set of rapidly growing solutions whose first component for large t_0 admits the estimate*

$$u_1(t) \le \ell\bigg(\int\limits_t^{+\infty} \big(p_1(s)p_2^{\lambda_1}(s)\big)^{\frac{1}{1+\lambda_1}}\,ds\bigg)^{-\frac{1}{\lambda}} \ \text{for } t \ge t_0,$$

where

$$\lambda = \frac{\lambda_1\lambda_2 - 1}{1+\lambda_1}, \ \ \ell = (1+\lambda_1)^{-\frac{1}{1+\lambda_1}}(1+\lambda_2)^{-\frac{\lambda_1}{1+\lambda_1}}(\lambda_1\lambda_2 - 1)^{-1}.$$

References

[1] I. V. Astashova, On Kiguradze's problem on power-law asymptotic behavior of blow-up solutions to Emden–Fowler type differential equations. *Georgian Math. J.* **24** (2017), no. 2, 185–191.

[2] R. Bellman, *Stability Theory of Differential Equations.* McGraw-Hill, New York, 1953.

[3] V. M. Evtukhov and A. V. Drozhzhina, Asymptotic representations of the solutions of nonautonomous ordinary differential equations. (Russian) *Ukraïn. Mat. Zh.* **71** (2019), no. 12, 1626–1646; translation in *Ukrainian Math. J.* **71** (2020), no. 12, 1865–1887.

[4] V. M. Evtukhov and A. M. Samoĭlenko, Asymptotic representations of solutions of nonautonomous ordinary differential equations with regularly varying nonlinearities. (Russian) *Differ. Uravn.* **47** (2011), no. 5, 628–650; translation in *Differ. Equ.* **47** (2011), no. 5, 627–649.

[5] N. A. Izobov, The Emden–Fowler equations with unbounded infinitely continuable solutions. (Russian) *Mat. Zametki* **35** (1984), no. 2, 189–199.

[6] N. A. Izobov, Kneser solutions. (Russian) *Differentsial'nye Uravneniya* **21** (1985), no. 4, 581–588.

[7] N. A. Izobov and V. A. Rabtsevich, The unimprovability of the I. T. Kiguradze–G. G. Kvinikadze condition for the existence of unbounded regular solutions of the Emden–Fowler equation. (Russian) *Differentsialnye Uravneniya* **23** (1987), no. 11, 1872–1881.

[8] I. T. Kiguradze, Asymptotic properties of solutions of a nonlinear differential equation of Emden–Fowler type. (Russian) *Izv. Akad. Nauk SSSR. Ser. Mat.* **29** (1965), no. 5, 965–986.

[9] I. T. Kiguradze, On the monotone solutions of nonlinear n-th order ordinary differential equations. (Russian) *Dokl. Akad. Nauk SSSR* **181** (1968), no. 5, 1054–1057; translation in *Sov. Math., Dokl.* **10** (1968), 980–983.

[10] I. T. Kiguradze, Monotone solutions of nonlinear ordinary differential equations of order n. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **33** (1969), 1373–1398; translation in *Math. USSR, Izv.* **3** (1969), 1293–1317.

[11] I. T. Kiguradze, On asymptotic behavior of solutions of nonlinear nonautonomous ordinary differential equations. *Qualitative theory of differential equations, Vol. I, II (Szeged, 1979)*, pp. 507–554, Colloq. Math. Soc. János Bolyai, 30, *North-Holland, Amsterdam–New York*, 1981.

[12] I. T. Kiguradze, Blow-up Kneser solutions of higher-order nonlinear differential equations. (Russian) *Differ. Uravn.* **37** (2001), no. 6, 735–743; translation in *Differ. Equ.* **37** (2001), no. 6, 768–777.

[13] I. T. Kiguradze and T. A. Chanturia, *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations.* Kluwer Academic Publishers, Dordrecht–Boston–London, 1993.

[14] I. T. Kiguradze and G. G. Kvinikadze, On strongly increasing solutions of nonlinear ordinary differential equations. *Ann. Math. Pura Appl.* **130** (1982), 67–87.

[15] J. D. Mirzov, *Asymptotic Properties of Solutions of Systems of Nonlinear Nonautonomous Ordinary Differential Equations.* Masaryk University, Brno, 2004.