On Admissible Perturbations of 3D Autonomous Polynomial ODE Systems

E. Musafirov, A. Grin, A. Pranevich

Yanka Kupala State University of Grodno, Grodno, Belarus

E-mails: musafirov@bk.ru; grin@grsu.by; pranevich@grsu.by

1 Introduction

In paper [3] V. I. Mironenko introduced the concept of a reflecting function to study the qualitative behavior of solutions of ODE systems. This function is now known as the *Mironenko reflecting function* (MRF) and has been successfully used to solve many problems in qualitative theory of ODE [1,4–7,13,14].

ODE systems with the same MRF have the same translation operator (see [2]) on any interval $(-\beta,\ \beta)$, and 2ω-periodic ODE systems with the same MRF have the same mapping on the period $[-\omega,\omega]$ (Poincare mapping). Therefore, some qualitative properties (such as the existence of periodic solutions and their stability) of solutions of ODE systems that have the same MRF are common. Thus, the study of the qualitative properties of solutions of a whole class of systems with the same MRF can be reduced to the corresponding study of a simple (well-studied) system. In such cases, non-autonomous systems can be studied on the basis of corresponding autonomous systems. In other words, some (well-studied) autonomous system can be transformed into a non-autonomous one with the help of special perturbations that preserve the MRF, which are called *admissible perturbations*. This provides new chances for researchers when modeling real-world processes and exploring novel (unstudied) ODE systems.

To search for admissible perturbations, we can use Theorem 1 from [5], which we formulate here in the form of the following lemma.

Lemma 1.1. *If the vector functions $\Delta_i(t,x)$ ($i=\overline{1,m}$, where $m\in\mathbb{N}$ or $m=\infty$) satisfy the identity*

$$\frac{\partial\Delta_i(t,x)}{\partial t}+\frac{\partial\Delta_i(t,x)}{\partial x}X(t,x)-\frac{\partial X(t,x)}{\partial x}\,\Delta_i(t,x)\equiv 0, \tag{1.1}$$

then the systems $\dot x = X(t,x)$ and $\dot x = X(t,x)+\sum\limits_{i=1}^{m}\alpha_i(t)\Delta_i(t,x)$ have identical MRF, where $t\in\mathbb{R}$, $x=(x_1,x_2,\dots,x_n)\in D\subset\mathbb{R}^n$, $\alpha_i(t)$ – arbitrary continuous scalar odd functions.

As initial systems, we consider well-known autonomous polynomial ODE systems (i.e. systems whose right-hand side $X(t,x)\equiv X(x)$, as well as the components of $X(x)$ are polynomials). The search for admissible perturbations is carried out by the method of undetermined coefficients, using identity (1.1) for vector functions $\Delta_i(t,x)\equiv\Delta_i(x)$ whose components are polynomials. That is, in this case, identity (1.1) is transformed to the form

$$\frac{\partial\Delta_i(x)}{\partial x}\,X(x)\equiv\frac{\partial X(x)}{\partial x}\,\Delta_i(x).$$

2 Examples of admissibly perturbed systems and their studies

Using Lemma 1.1 and the approach outlined above, for the Hindmarsh–Rose neuron model

$$\begin{aligned} \dot{x} &= y - ax^3 + bx^2 - z + I, \\ \dot{y} &= c - dx^2 - y, \qquad\qquad x, y, z, a, b, c, d, I, r, s, \alpha \in \mathbb{R} \\ \dot{z} &= r(s(x-\alpha) - z), \end{aligned}$$

admissible perturbations are obtained in [11]. Numerical examples show that admissibly perturbed systems have similar bifurcation diagrams, periodic attractors and strange attractor as the original Hindmarsh–Rose system.

In [10] admissible perturbations are obtained for the Lorentz-84 system, which models the general circulation of the atmosphere in mid-latitudes:

$$\begin{aligned} \dot{x} &= -ax - y^2 - z^2 + aF, \\ \dot{y} &= -y + xy - bxz + G, \qquad a, b, F, G, x, y, z \in \mathbb{R}. \\ \dot{z} &= -z + bxy + xz, \end{aligned} \tag{2.1}$$

In particular, it has been proven that the MRF of system (2.1) and the system

$$\begin{aligned} \dot{x} &= (-ax - y^2 - z^2 + aF)(1 + \alpha_1(t)), \\ \dot{y} &= (-y + xy - bxz)(1 + \alpha_1(t)) - z\alpha_2(t), \\ \dot{z} &= (-z + bxy + xz)(1 + \alpha_1(t)) + y\alpha_2(t) \end{aligned} \tag{2.2}$$

coincide if $G = 0$ and $\alpha_i(t)$ are arbitrary continuous scalar odd functions ($i = \overline{1,2}$). The results of the analysis of the qualitative behavior of solutions of the original system (2.1) are extended to the perturbed system (2.2) and the following theorem is proved.

Theorem 2.1. *Suppose that $\alpha_i = \alpha_i(t)$ ($i = \overline{1,2}$) are continuous functions (not necessarily odd). Then the following statements hold:*

(1) *if $a > 0$, $F < 1$ and $\alpha_1(t) \geqslant c > -1$ $\forall\, t \geqslant 0$ (c is a constant), then the equilibrium solution $x = F$, $y = z = 0$ of system (2.2) is globally exponentially stable (exponentially stable in the large);*

(2) *if $a \geqslant 0$, $F \leqslant 1$ and $\alpha_1(t) \geqslant -1$ $\forall\, t \geqslant 0$, then the equilibrium solution $x = F$, $y = z = 0$ of system (2.2) is globally uniformly Lipschitz stable;*

(3) *if $a > 0$, $F > 1$ and $\alpha_1(t) \geqslant c > -1$ $\forall\, t \geqslant 0$ (c is a constant), then the equilibrium solution $x = F$, $y = z = 0$ of system (2.2) is Lyapunov unstable.*

For the Langford system, which models turbulence in a liquid, presented in the form (more often found in Russian-language literature):

$$\begin{aligned} \dot{x} &= (2a-1)x - y + xz, \\ \dot{y} &= x + (2a-1)y + yz, \qquad a, x, y, z \in \mathbb{R}, \\ \dot{z} &= -az - (x^2 + y^2 + z^2), \end{aligned}$$

admissible perturbations are obtained in [8]. And for the Langford system, presented in the form:

$$\begin{aligned} \dot{x} &= (a-1)x - y + xz, \\ \dot{y} &= x + (a-1)y + yz, \qquad a, x, y, z \in \mathbb{R}, \\ \dot{z} &= az - (x^2 + y^2 + z^2), \end{aligned}$$

admissible perturbations are obtained in [9].

In [12], admissible perturbations are obtained for the generalized Langford system

$$\begin{aligned} \dot{x} &= ax + by + xz, \\ \dot{y} &= cx + dy + yz, \qquad\qquad a, b, c, d, e, x, y, z \in \mathbb{R}. \\ \dot{z} &= ez - (x^2 + y^2 + z^2), \end{aligned} \tag{2.3}$$

In particular, it has been proven that the MRF of system (2.3) and the system

$$\begin{aligned} \dot{x} &= (ax + by + xz)(1 + \alpha_1(t)) + x(a + z)\alpha_2(t) + y\alpha_3(t) \\ &\qquad - y(x^2 + y^2)(4az + x^2 + y^2 + 2z^2)\alpha_4(t), \\ \dot{y} &= (-bx + ay + yz)(1 + \alpha_1(t)) + y(a + z)\alpha_2(t) - x\alpha_3(t) \\ &\qquad + x(x^2 + y^2)(4az + x^2 + y^2 + 2z^2)\alpha_4(t), \\ \dot{z} &= -(2az + x^2 + y^2 + z^2)(1 + \alpha_1(t) + \alpha_2(t)) \end{aligned} \tag{2.4}$$

coincide if $c = -b$, $d = a$, $e = -2a$ and $\alpha_i(t)$ are arbitrary continuous scalar odd functions ($i = \overline{1,4}$). The obtained result allows us to extend the results of the analysis of the qualitative behavior of solutions of the original system (2.3) to solutions of the perturbed system (2.4). In particular, the following statements are proven in [12].

Theorem 2.2. *Let $\alpha_i(t)$ $(i = \overline{1,4})$ be scalar continuous functions (not necessarily odd).*

(1) *If $a = 0$ and $\alpha_1(t) + \alpha_2(t) \geqslant l > -1$ $\forall\, t \geqslant 0$ ($l = const$), then the solution $x = y = z = 0$ of system (2.4) is Lyapunov unstable.*

(2) *If $b = 0$ and the function $\alpha_3(t) + a^4\alpha_4(t)$ is ω-periodic and $\exists\, k \in \mathbb{Z}$ such that $\int\limits_0^\omega (\alpha_3(s) + a^4\alpha_4(s))\,\mathrm{d}s = 2\pi k$, then the solution*

$$\begin{aligned} x(t) &= a \sin\left(bt + \int\limits_0^t (b\alpha_1(s) + \alpha_3(s) + a^4\alpha_4(s))\,\mathrm{d}s\right), \\ y(t) &= a \cos\left(bt + \int\limits_0^t (b\alpha_1(s) + \alpha_3(s) + a^4\alpha_4(s))\,\mathrm{d}s\right), \\ z(t) &= -a \end{aligned} \tag{2.5}$$

of system (2.4) is ω-periodic (the period is not necessarily minimal).

(3) *If $b \neq 0$ and the function $b\alpha_1(t) + \alpha_3(t) + a^4\alpha_4(t)$ is $2\pi/\,|b|$-periodic and $\int\limits_0^{2\pi/b} (b\alpha_1(s) + \alpha_3(s) + a^4\alpha_4(s))\,\mathrm{d}s = 0$, then solution (2.5) of system (2.4) is $2\pi/|b|$-periodic (the period is not necessarily minimal).*

Theorem 2.3. *Let $\alpha_i(t)$ $(i = \overline{1,4})$ be scalar twice continuously differentiable odd functions, $b \neq 0$ and the right side of system (2.4) be $2\pi/|b|$-periodic in t. If $\exists\, k \in \mathbb{Z}$ such that $\int\limits_0^{-2\pi/|b|} (b\alpha_1(s) + \alpha_3(s) + a^4\alpha_4(s))\,\mathrm{d}s = 2\pi k$, then solution (2.5) of system (2.4) is $2\pi/|b|$-periodic.*

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