Pointwise Conditions of Solvability of a Periodic Problem for Higher Order Functional Differential Equations

Sulkhan Mukhigulashvili

Institute of Mathematics, Academy of Sciences of the Czech Republic
Brno, Czech Republic
E-mail: smukhig@gmail.com

Abstract

On the interval $I := [a, b]$, we study the higher order linear functional-differential equation

$$u^{(n)}(t) = \ell(u)(t) + q(t), \tag{1}$$

where $q \in L(I; R)$, $\ell : C(I; R) \to L^\infty(I; R)$ is a linear bounded operator, under the periodic type boundary conditions

$$u^{(i)}(\omega) - u^{(i)}(0) = c_i \ \ (i = 0, \dots, n-1). \tag{2}$$

The obtained pointwise efficient sufficient conditions of unique solvability of our problem are non-improvable and moreover, for such important classes of functional-differential equations as differential or integro-differential equations with deviated argument are these conditions take into account the effect of argument deviation and generalize some previously known results (see, for example, [1–3]). Also on the basis of the mentioned results for the linear problem, there are proved non-improvable efficient sufficient conditions of solvability of the periodic type problem for the nonlinear functional differential equation

$$u^{(n)}(t) = F(u)(t) + f_0(t) \ \ \text{for} \ \ t \in [0, \omega], \tag{3}$$

where $F : C(I; R) \to L(I; R)$ is a Carathéodory's local class operator and $f_0 \in L(I, R)$.

Main results

Let $m \in N$, $\sigma \in \{-1, 1\}$, and consider the numbers defined by the following equations

$$\gamma_{n,\sigma} = \begin{cases} 1 & \text{for} \ \ n = 2m, \ \ \sigma = (-1)^m, \\ 0 & \text{for} \ \ n = 2m, \ \ \sigma = (-1)^{m+1}, \\ 0 & \text{for} \ \ n = 2m+1, \ \ \sigma \in \{-1, 1\}. \end{cases}$$

Let also for an arbitrary $x \in I = [0, \omega]$ and a monotone linear operator ℓ, the nonnegative functions $\Delta_x \in C(I; R_0^+)$, and $\rho_\ell \in L^\infty(I; R_0^+)$ be defined by the equalities

$$\Delta_x(t) = |t - x|, \quad \rho_\ell(t) = \frac{2\pi}{\omega} \Big(\ell(1)(t) \int\limits_0^\omega \ell(\Delta_s)(s)\, ds \Big)^{1/2}.$$

Then the following theorem is true.

Theorem 1. *Let $\sigma \in \{-1, 1\}$, and the monotone linear operator $\ell : C(I;R) \to L^\infty(I;R)$ satisfy the conditions*

$$\sigma \int\limits_0^\omega \ell(1)(s)\,ds > 0,$$

and

$$\gamma_{n,\sigma}|\ell(1)(t)| + \rho_\ell(t) < \Big(\frac{2\pi}{\omega}\Big)^n \ \ \text{for} \ \ t \in I.$$

Then problem $(1),(2)$ *is uniquely solvable.*

Due to the definition of the constant $\gamma_{n,\sigma}$ from our theorem it immediately follows

Corollary 1. *Let $m \in N$, $\ell : C(I;R) \to L^\infty(I;R)$ be the monotone linear operator, and*

$$n = 2m+1 \ \ \text{and} \ \ \int\limits_0^\omega \ell(1)(s)\,ds \neq 0,$$

or

$$n = 2m \ \ \text{and} \ \ (-1)^{m+1} \int\limits_0^\omega \ell(1)(s)\,ds > 0.$$

Then the condition

$$\ell(1)(t) \int\limits_0^\omega \ell(\Delta_s)(s)\,ds < \Big(\frac{2\pi}{\omega}\Big)^{2(n-1)} \ \ \text{for} \ \ t \in I$$

guarantees the unique solvability of problem $(1),(2)$.

Now assume that $\ell(u)(t) = p(t)u(t)$, where $p \in L^\infty(I;R)$, i.e. we assume that (1) is the ordinary differential equation

$$u^{(n)}(t) = p(t)u(t) + q(t) \ \ \text{for} \ \ t \in I. \tag{4}$$

Then it is clear that $\ell(\Delta_t)(t) = p(t)|t-t| \equiv 0$, and therefore from our theorem it follows:

Corollary 2. *Let $\sigma \in \{-1,1\}$, and a constant sign function $p \in L^\infty(I;R)$ satisfy the conditions*

$$\sigma \int\limits_0^\omega p(s)\,ds > 0 \ \ \text{and} \ \ \gamma_{n,\sigma}|p(t)| < \Big(\frac{2\pi}{\omega}\Big)^n \ \ \text{for} \ \ t \in I.$$

Then problem $(4),(2)$ *is uniquely solvable.*

But this proposition is I. Kiguradze and T. Kusano's theorem from [1], and there was shown that $(\frac{2\pi}{\omega})^n$ is optimal.

Now we consider the nonlinear problem $(3),(2)$. To formulate the main theorem we need the following definition.

Definition. Let $\sigma \in \{-1,1\}$. We will say that the operator $h : C(I;R) \to L^\infty(I;R)$ belongs to the class $\mathcal{K}_\omega^{\sigma,n}$ if h is a nonnegative linear operator,

$$h(1)(t) \not\equiv 0,$$

and for an arbitrary $\alpha \in L^\infty(I;R)$ such that

$$\alpha \not\equiv 0, \ \ 0 \le \alpha(t) \le 1 \ \ \text{for} \ \ t \in I,$$

the homogeneous problem

$$v^{(n)}(t) = \sigma\alpha(t)h(v)(t) \ \text{ for } \ t \in I,$$
$$v^{(i)}(\omega) - v^{(i)}(0) = 0 \ \ (i = 0, \dots, n-1)$$

has no nontrivial solution.

Note that in the given theorem the function $\eta : I \times R_0^+ \to R_0^+$ is summable in the first argument, nondecreasing in the second one, and satisfies the condition

$$\lim_{\rho \to +\infty} \frac{1}{\rho} \int_0^{\omega} \eta(s, \rho)\, ds = 0.$$

Theorem 2. *Let the linear nonnegative operator $h : C(I;R) \to L^{\infty}(I;R)$, the function $g_0 \in L(I;R)$, and numbers $\sigma \in \{-1, 1\}$, $r_0 > 0$ be such that the condition*

$$g_0(t) \le \sigma F(x)(t) \operatorname{sign} h(x)(t) \le |h(x)(t)| + \eta(t, \, ||x||_{C^{n-1}}) \ \ \text{if} \ \ ||x||_{C^{n-1}} \ge r_0,$$

on I, and the inclusion

$$h \in \mathcal{K}_{\omega}^{\sigma,n}$$

hold. Moreover, let $g \in L(I;R)$ be such that on I the condition

$$g(t) \le \sigma F(x)(t) \operatorname{sign} h(x)(t) \ \ \text{if} \ \ \min_{t \in I} |x(t)| \ge r_0$$

is fulfilled, and

$$\int_0^{\omega} g(s)\, ds - \left| \int_0^{\omega} f_0(s)\, ds \right| \ge |c_{n-1}|.$$

Then problem $(3), (2)$ *has at least one solution.*

Now we give a corollary of our theorem for the following ordinary differential equation

$$u''(t) = f(t, x(\tau(t))) + f_0(t) \ \text{ for } \ t \in I. \tag{5}$$

Corollary 3. *Let numbers $\sigma \in \{-1, 1\}$, $r_0 > 0$, functions $h \in L^{\infty}(I;R)$, $g_0 \in L(I;R)$, and a measurable function $\tau : I \to I$ be such that conditions*

$$\gamma_{\sigma,n} h(t) + \rho_h(t) < \left(\frac{2\pi}{\omega}\right)^n \ \ \text{for} \ \ t \in I,$$
$$g_0(t) \le \sigma f(t, x) \operatorname{sign} x \le h(t)|x| + \eta(t, |x|) \ \ \text{for} \ \ |x| \ge r_0, \ \ t \in I,$$

and

$$\int_0^{\omega} g_0(s)\, ds - \left| \int_0^{\omega} f_0(s)\, ds \right| \ge |c_{n-1}|$$

hold. Then problem $(5), (2)$ *has at least one solution.*

References

[1] I. T. Kiguradze and T. Kusano, On periodic solutions of higher-order nonautonomous ordinary differential equations. (Russian) *Differ. Uravn.* **35** (1999), no. 1, 72–78; translation in *Differential Equations* **35** (1999), no. 1, 71–77.

[2] S. Mukhigulashvi and V. Novotná, Some two-point problems for second order integro-differential equations with argument deviations. *Topol. Methods Nonlinear Anal.* **54** (2019), no. 2A, 459–476.

[3] S. Mukhigulashvili, N. Partsvania and B. Půža, On a periodic problem for higher-order differential equations with a deviating argument. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods* **74** (2011), no. 10, 3232–3241.