Sturm–Liouville Operators with Strongly Singular Coefficients: Semi-Boundedness and Self-Adjointness

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The problem of symmetric operators being self-adjoint is one of the main problems in theory of differential operators and serves as the basis for the analysis of their spectral properties and scattering problems. Investigation of this problem for Sturm–Liouville and Schrödinger operators in the space $L^2(\mathbb{R})$ is inspired by the problems of mathematical physics and has numerous applications. The results of that obtained for this problem in the case of regular coefficients are rather complete.

We introduce and investigate symmetric operators L_0 associated in the complex Hilbert space $L^2(\mathbb{R})$ with a formal differential expression

$$l[u] := -(pu')' + qu + i((ru)' + ru') \tag{1}$$

under minimal conditions on the regularity of the coefficients. They are assumed to satisfy conditions

$$q = Q' + s; \quad \frac{1}{\sqrt{|p|}}, \frac{Q}{\sqrt{|p|}}, \frac{r}{\sqrt{|p|}} \in L^2_{loc}(\mathbb{R}), \ \ s \in L^1_{loc}(\mathbb{R}), \tag{2}$$

where the derivative of the function Q is understood in the sense of distributions, and all functions p, Q, r, s are real-valued. In particular, the coefficients q and r' may be Radon measures on $\mathbb{R}$, while function p may be discontinuous. Our main results are two sufficient conditions on coefficients p which provide that the operator L_0 being semi-bounded implies it being self-adjoint.

If these coefficients of (1) are regular enough, then the mapping

$$\mathrm{L}_{00} : u \mapsto l[u], \ \ u \in C_0^\infty(\mathbb{R})$$

defines a densely defined in the complex Hilbert space $L^2(\mathbb{R})$ preminimal symmetric operator L_{00}. Here naturally arises question whether the closure of this operator $\mathrm{L}_0 := (\mathrm{L}_{00})^\sim$ is self-adjoint. A large number of papers are devoted to this problem (see, e.g. the references in [12]). For instance, Hartman [5] and Rellich [10] established that if operator L_{00} is bounded from below and

$$r \equiv 0, \quad 0 < p \in C^2(\mathbb{R}), \ \ q \text{ is piecewise continuous on } \mathbb{R},$$

and function p satisfies the condition

$$\int\limits_0^\infty p^{-1/2}(t)\,dt = \int\limits_{-\infty}^0 p^{-1/2}(t)\,dt = \infty, \tag{3}$$

then the minimal operator L_0 corresponding to l is self-adjoint. In the paper [11], the conditions on the regularity of the coefficients of l were weakened:

$$r \equiv 0, \quad 0 < p \ \text{ is locally Lipschitz}, \ \ q \in L^2_{loc}(\mathbb{R}).$$

Another sufficient condition for the operator L_0 to be self-adjoint was obtained in [1]. It may be written in the form

$$\|p\|_{L^\infty(-\rho,-\rho/2)},\ \ \|p\|_{L^\infty(\rho/2,\rho)} = O(\rho^2),\ \ \rho\to\infty. \tag{4}$$

Here the coefficients of (1) satisfy the conditions

$$r\equiv 0,\quad 0<p\in W^1_{2,loc}(\mathbb{R}),\ \ q\in L^1_{loc}(\mathbb{R}).$$

Examples show that conditions (3) and (4) are independent (see [1]).

We propose to consider the operators generated by the formal differential expression (1) as quasi-differential operators, which are defined applying compositions of differential operators with locally summable coefficients. These operators are defined using the Shin–Zettl matrix function specifically chosen to correspond to the coefficients of l (see [2–4, 13]).

In our case it has the form

$$A(x)=\begin{pmatrix} \dfrac{Q+ir}{p} & \dfrac{1}{p} \\ -\dfrac{Q^2+r^2}{p}+s & -\dfrac{Q-ir}{p}\end{pmatrix}$$

and, due to our assumptions, belongs to the class $L^1_{loc}(\mathbb{R},\mathbb{C}^2)$.

It can be used to define corresponding quasi-derivatives as follows:

$$\begin{aligned} u^{[0]} &:= u,\\ u^{[1]} &:= pu'-(Q+ir)u,\\ u^{[2]} &:= (u^{[1]})'+\frac{Q-ir}{p}\,u^{[1]}+\Big(\frac{Q^2+r^2}{p}-s\Big)u. \end{aligned}$$

A formal differential expression (1) may now be defined as quasi-differential:

$$l[u]:=-u^{[2]},\quad \mathrm{Dom}(l):=\big\{u:\mathbb{R}\to\mathbb{C}\ |\ \ u,u^{[1]}\in \mathrm{AC}_{loc}(\mathbb{R})\big\}.$$

This definition is motivated by the fact that

$$\langle -u^{[2]},\varphi\rangle=\big\langle -(pu')'+qu+i((ru)'+ru'),\varphi\big\rangle\quad \forall\,\varphi\in \mathrm{C}_0^\infty(\mathbb{R})$$

in the sense of distributions.

We define for the quasi-differential expression l the operators L and L_{00} as:

$$\begin{gathered} \mathrm{L}u:=l[u],\quad \mathrm{Dom}(\mathrm{L}):=\big\{u\in L^2(\mathbb{R})\ |\ \ u,u^{[1]}\in \mathrm{AC}_{loc}(\mathbb{R}),\ l[u]\in L^2(\mathbb{R})\big\},\\ \mathrm{L}_{00}u:=\mathrm{L}u,\quad \mathrm{Dom}(\mathrm{L}_{00}):=\big\{u\in \mathrm{Dom}(\mathrm{L})\ |\ \ \operatorname{supp}u\Subset\mathbb{R}\big\}. \end{gathered}$$

The operators L and L_{00} are maximal and preminimal operators for expression l, respectively. Their definitions coincide with the classical ones if the coefficients l are sufficiently smooth. It can be shown that the operator L_{00} is densely defined in $L^2(\mathbb{R})$ and is symmetric.

Let us formulate the main results of the paper in the form of two theorems. The first of them is a natural generalization of the above-mentioned result of Hartman and Rellich.

Theorem 1. *Let the coefficients of the formal differential expression* (1) *satisfy the assumptions* (2) *and also*

(i) $p\in W^1_{2,loc}(\mathbb{R})$, $p>0$,

(ii) $\int\limits_{-\infty}^{0} p^{-1/2}(t)\,dt = \int\limits_{0}^{\infty} p^{-1/2}(t)\,dt = \infty.$

Then, if operator L_{00} *is bounded from below, then it is essentially self-adjoint and* $\mathrm{L}_{00}^{*} = \mathrm{L} = \mathrm{L}^{*}$.

For the case $p \equiv 1$, $r \equiv 0$, Theorem 1 was previously established in [6].

In the second theorem, additional conditions on the coefficient p are imposed not on the entire axis, but only on a sequence of finite intervals. However, outside of these intervals the function p may vanish and be discontinuous.

Theorem 2. *Suppose the assumptions* (2) *are satisfied and the operator* L_{00} *is bounded from below. Suppose the sequence of intervals* $\Delta_n := [a_n, b_n]$ *exists such that*

$$-\infty < a_n < b_n < \infty, \quad b_n \to -\infty,\ \ n \to -\infty, \quad a_n \to \infty,\ \ n \to \infty,$$

where the coefficients p satisfy the additional conditions:

(i) $p_n := p|_{\Delta_n} \in W_2^1(\Delta_n)$, $p_n > 0$;

(ii) $\exists\, C > 0 :\ p_n(x) \leq C|\Delta_n|^2$, $n \in \mathbb{Z}$, *where* $|\Delta_n|$ *is the length of interval* Δ_n.

Then operator L_{00} *is essentially self-adjoint and* $\mathrm{L}_{00}^{*} = \mathrm{L} = \mathrm{L}^{*}$.

For the case $p \equiv 1$, $r \equiv 0$, necessary and sufficient conditions for semi-boundedness of operator L_{00} were obtained in [8].

The proofs of Theorem 1 and Theorem 2 can be found in [9].

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