On the Stability of Toroidal Manifold for One Class of Dynamical System

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We study the stability of the invariant toroidal manifold of one class, the linear extension of a dynamical system on a torus. The result is used to study the question of the existence of an invariant manifold of a nonlinear system of differential equations.

In the direct product of an m-dimensional torus T_m and Euclidean space R^n we consider a system of differential equations

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dx}{dt} = P(\varphi)x, \tag{1}$$

where $\varphi = col(\varphi_1, \dots, \varphi_m)$, $x = col(x_1, \dots, x_n)$, $a(\varphi), P(\varphi)$ are, respectively, vector and matrix functions continuous and 2π-periodic in each component φ_j $(j = 1, \dots, m)$ and defined on the m-dimensional torus T_m. Assume that the function $a(\varphi)$ satisfies the Lipschitz condition with respect to φ, a constant L, i.e. for any two points $\varphi', \varphi'' \in T_m$ we have

$$\|a(\varphi') - a(\varphi'')\| \le L\|\varphi' - \varphi''\|. \tag{2}$$

We establish sufficient conditions for the asymptotic stability of the trivial torus of system (1) and use these results for the investigation of nonlinear system of differential equations more complicated than system (1) and defined in the direct product $T_m \times R^n$.

In what follows, we need a generalization of the Wazewski inequality [2]. By $\varphi_t(\varphi)$ we denote the solution of the first equation in system (1) and consider a system of equation

$$\frac{dx}{dt} = P(\varphi_t(\varphi))x \tag{2}$$

for x. According to the Wazewski theorem [2], any solution $x_t(t_0, \varphi, x_0)$, $x_{t_0}(t_0, \varphi, x_0) = x_0$ of this system admits

$$\|x_0\| e^{\int\limits_{t_0}^{t} \lambda(\varphi_s(\varphi))\,ds} \le \|x_t(t_0, \varphi, x_0)\| \le \|x_0\| e^{\int\limits_{t_0}^{t} \Lambda(\varphi_s(\varphi))\,ds}, \tag{3}$$

where $\lambda(\varphi)$ and $\Lambda(\varphi)$ are, respectively, the maximum and minimum eigenvalues of the symmetric matrix

$$\widehat{P}(\varphi) = \frac{1}{2}\left(P(\varphi) + P^T(\varphi)\right),$$

$P^T(\varphi)$ is the matrix transposed with respect to the matrix $P(\varphi)$.

Inequality (3) yields the estimate

$$\|x_0\|e^{\lambda(t-t_0)} \le \|x_t(t_0, \varphi, x_0)\| \le \|x_0\|e^{\Lambda(t-t_0)}, \ \ t \ge t_0,$$

where

$$\lambda = \min_{\varphi \in T_m} \lambda(\varphi), \quad \Lambda = \max_{\varphi \in T_m} \Lambda(\varphi).$$

On the basic of this estimate, we can make the following conclusion: If the matrix $P(\varphi)$ in system (1) is such that $\Lambda < 0$, than the nontrivial torus of this system is exponentially stable because the matricant $\Omega^t_{t_0}(\varphi)$ of system (2) admits the estimate

$$\|\Omega^t_{t_0}(\varphi)\| \le Ke^{-\gamma(t-t_0)}, \ \ t \ge t_0, \tag{4}$$

$\varphi \in T_m$, $K \ge 1$, $\gamma > 0$.

We now show that a similar conclusion concerning the exponential stability of the trivial torus of the system of equations (1) can be made under weaker conditions imposed on the matrix $P(\varphi)$.

Recall [5] that a point $\varphi \in T_m$ of the dynamical system on a torus

$$\frac{d\varphi}{dt} = a(\varphi) \tag{5}$$

is called wandering if there exist its neighborhood $U(\varphi)$ and a positive number T such that

$$U(\varphi) \cap \varphi_t(U(\varphi)) = \varnothing \ \text{ for } \ t \ge T.$$

By W we denote the set of wandering points and by $\Omega = T_m \setminus W$ we denote the set of nonwandering points. The set W of wandering points is invariant and open because, together with φ, all points of the neighborhood $U(\varphi)$ are wandering.

In view of the compactness of the torus T_m, the set of nonwandering point Ω is a nonempty closed invariant set.

It is clear that Ω is also a compact set as a closed set on the torus.

As shown in [5], any solution of system (5) eventually approaches the set of nonwandering points. More precisely, for any $\varepsilon > 0$, every phase point $\varphi_t(\varphi)$ lies outside the ε-neihghborhood $U_\varepsilon(\Omega)$ of the set Ω only for a finite time interval not larger than $T(\varepsilon)$.

To prove the theorem presented below, we use the property of nonwandering points.

Theorem 1. *If the matrix* $P(\varphi)$ *in the system of equations* (1) *such that the maximum eigenvalue* $\Lambda(\varphi)$ *of the symetric matrix* $\widehat{P}(\varphi)$ *is negative on the set* Ω *of nonwandering points of the dynamical system* (5), *then the trivial torus of system* (1) *is exponentially stable.*

Proof. We fix sufficiently small ε-neighborhood $U_\varepsilon(\Omega)$ of the set Ω. Since $\Lambda(\varphi) < 0$ for all $\varphi \in \Omega$ and Ω is closed compact set, one can find a sufficiently small positive number ε_0 such that $\Lambda(\varphi) < -\gamma(\varepsilon)$ for any $0 \le \varepsilon \le \varepsilon_0$, $\Lambda(\varphi) < -\gamma(\varepsilon)$ and all $\varphi \in U_\varepsilon(\Omega)$, where $\gamma(\varepsilon)$ is a positive monotonically nonincreasing function of the parameter ε such that $\gamma(\varepsilon) \to \gamma(0)$ as $\varepsilon \to 0$, where

$$-\gamma(0) = \max_{\varphi \in \Omega} \Lambda(\varphi).$$

If $\varphi_t(\varphi)$ is nonwandering trajectory, then, for any solution from inequality (3), we get the following estimate:

$$\|x_t(t_0, \varphi, x_0)\| \le \|x_0\|e^{-\gamma(0)(t-t_0)}, \ \ t \ge t_0, \ \ \varphi \in \Omega.$$

If $\varphi_t(\varphi)$ is a wandering trajectory, then one can find a positive number $T(\varepsilon)$ such that the time of stay of this trajectory outside the set $U_\varepsilon(\Omega)$ is not greater than $T(\varepsilon)$. By using inequality (3), we obtain

$$\|x_t(t_0,\varphi,x_0)\| \le \|x_0\| e^{\Lambda T(\varepsilon)} \cdot e^{-\gamma(\varepsilon)(t-t_0)}, \ \ t \ge t_0, \ \ \varphi \in W.$$

Hence, under the conditions of the theorem, any solution $x_t(t_0,\varphi,x_0)$ of system (2) exponentially approaches zero as $t \to \infty$ for any $\varphi \in T_m$. Therefore, the matricant $\Omega^t_{t_0}(\varphi)$ of the system admits an estimate of the form (4), which completes the proof of the theorem. □

We now present one more class of system (1) for which the trivial torus is asymptotically stable.

Theorem 2. *If the matrix function $P(\varphi)$ in the system of equations* (1) *satisfies the condition*

$$\langle P(\varphi)x, x\rangle \le \gamma(\varphi)\langle x, x\rangle$$

for all $\varphi \in T_m$ and $x \in R^n$, where $\gamma(\varphi)$ is a function continuous and 2π-periodic in each component φ_j $(j = 1,\dots,n)$ and negative on the set Ω of nonwandering points of the dynamical system (5), *then the trivial torus of the original system* (1) *is asymptotically stable.*

Proof. For any solution $x_t(t_0,\varphi,x_0)$ of system (2), we obtain:

$$\begin{aligned}\frac{d}{dt}\,\|x_t(t_0,\varphi,x_0)\|^2 &= \frac{d}{dt}\,\big\langle x_t(t_0,\varphi,x_0), x_t(t_0,\varphi,x_0)\big\rangle \\ &= 2\big\langle P(\varphi_t(\varphi))x_t(t_0,\varphi,x_0), x_t(t_0,\varphi,x_0)\big\rangle \le 2\gamma(\varphi_t(\varphi))\|x_t(t_0,\varphi,x_0)\|^2.\end{aligned}$$

Integrating the last inequality, we find

$$\|x_t(t_0,\varphi,x_0)\| \le e^{\int\limits_{t_0}^{t} \gamma(\varphi_S(\varphi))\,dS}\,\|x_0\|, \ \ t \ge t_0, \ \ \varphi \in T_m.$$

Reasoning as in the proof of the previous theorem, we conclude that the exponential stability of the trivial torus of the original system follows from the last inequality. □

To prove that the invariant torus is stable (unstable), we can use the direct Lyapunov method. We now present a theorem that partially supplements the result of classical investigations in this field presented in the monographs [3, 7].

Theorem 3. *Suppose that, for the system of equations* (1)*, there exists a positive-definite quadratic form*

$$V(\varphi,x) = \langle S(\varphi)x, x\rangle$$

with symmetric matrix $S(\varphi)$ such that its total derivative composed with the use of the original system (1)*, i.e., the quadratic form*

$$\frac{d}{dt}V(\varphi,x) = \langle \widehat{S}(\varphi)x, x\rangle,$$

where

$$\widehat{S}(\varphi) = \frac{\partial S(\varphi)}{\partial \varphi}\cdot a(\varphi) + S(\varphi)P(\varphi) + P^T(\varphi)S(\varphi),$$

is negative-definite of the set Ω of nonwandering points of system (5)*. Then the trivial torus of the system of equations* (1) *is exponentially stable.*

It is natural to study the problem of existence of the quadratic form $V(\varphi, x)$, atisfying the conditions of Theorem 3.

We now present an example in which this form exists and makes it possible to state that the trivial torus of system (1) is exponentially stable.

Theorem 4. *Suppose that $P(\varphi)$ in system* (1) *is a constant matrix $P(\varphi) = P_0$ on the set Ω. If the real parts of the eigenvalues $\operatorname{Re}\lambda_j(P_0)$ of the matrix P_0 are negative, then there exists a positive-definite quadratic form $v(\varphi, x) = \langle S(\varphi)x, x\rangle$ with symmetric matrix $S(\varphi)$ such that its derivative, according to system* (1)*, is a negative-definite quadratic form on the set Ω, and, hence, the trivial torus of system* (1) *is asymptotically stable.*

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