Uncertainty Estimates for Target Functionals Values to a Class of Continuous-Discrete Systems with Incomplete Information

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1 Introduction

A linear control system with continuous and discrete times and discrete memory is considered. The model includes an uncertainty in the description of operators implementing control actions. This uncertainty is a consequence of random disturbances under the assumption of their uniform distribution over known intervals.

With each implementation a corresponding trajectory arises from random perturbations, and in the aggregate – an ensemble of trajectories. Thus, there arises a set of values of target functionals in the control problem. For each functional, the probabilistic description is given in the form of corresponding probability density functions. To construct these functions, the previously obtained representation of the Cauchy operator of the system under consideration is used. The proposed probabilistic description allows one to find their standard characteristics, including expectation and variance, as well as the entire possible range of values. The results are constructive in nature and allow for effective computer implementation.

2 Description of the problem

Fix a finite segment $[0,T]\subset\mathbb{R}$. Denote by $L^n=L^n[0,T]$ the space of summable functions $v:[0,T]\to\mathbb{R}^n$ with the norm $\|v\|_{L^n}=\int\limits_0^T|v(s)|_n\,ds$, where $|\cdot|_n$ stands for a norm in $\mathbb{R}^n$; $L_2^r=L_2^r[0,T]$ is the space of square summable functions $v:[0,T]\to\mathbb{R}^r$ with the inner product $\langle u,v\rangle=\int\limits_0^T u'(s)\cdot v(s)\,ds$ ($(\,\cdot\,)'$ stands for transposition); $AC^n=AC^n[0,T]$ is the space of absolutely continuous functions $x:[0,T]\to\mathbb{R}^n$ with the norm $\|x\|_{AC^n}=|x(0)|_n+\|\dot{x}\|_{L^n}$. Next we fix the set $J=\{t_0,t_1,\dots,t_\mu\}$, $0=t_0<t_1<\dots<t_\mu=T$ and denote by $FD^\nu(\mu)=FD^\nu\{t_0,t_1,\dots,t_\mu\}$ the space of functions of discrete argument $z:J\to\mathbb{R}^\nu$ with the norm

$$\|z\|_{FD^\nu(\mu)}=\sum_{i=0}^{\mu}|z(t_i)|_\nu.$$

We consider the continuous-discrete system with discrete memory

$$\dot{x}(t)=\sum_{j:\,t_j<t}A_j(t)x(t_j)+\sum_{j:\,t_j<t}B_j(t)z(t_j)+(Fu)(t),\ \ t\in[0,T], \tag{2.1}$$

$$z(t_i)=\sum_{j<i}D_{ij}x(t_j)+\sum_{j<i}H_{ij}z(t_j)+(Gu)(t_i),\ \ i=1,\dots,\mu. \tag{2.2}$$

Here the columns of $(n \times n)$-matrix A_j and $(n \times \nu)$-matrix B_j belong to L^n; D_{ij} and H_{ij} are constant matrices of dimension $(\nu \times n)$ and $(\nu \times \nu)$, respectively; $F : L_2^r \to L^n$, $G : L_2^r \to FD^\nu(\mu)$ are linear bounded Volterra [1] operators. Discreteness of the memory to all operators acting onto the state variable in (2.1), (2.2) is defined by their construction.

For the system under control (2.1), (2.2) with a given initial state

$$x(0) = \alpha, \quad z(0) = \delta \tag{2.3}$$

we consider the control problem with the aim of control given by the equality

$$\ell(x, z) = \beta \in \mathbb{R}^{\mathcal{N}}, \tag{2.4}$$

where $\ell : AC^n \times FD^\nu(\mu) \to \mathbb{R}^N$ is a linear bounded vector-functional.

Conditions of the solvability to the problem (2.1), (2.2) within the class of programmed control are obtained for the case of unconstrained control and for the case of point-wise polyhedral constraints [2,4,7,8]. Here we study the question on the impact of random disturbances of operators F and G onto the values of the target vector-functional $\ell(x, z)$ when the control is known. Without loss of generality we suppose the initial position of the system (2.1), (2.2) to be zero: $\alpha = 0$, $\delta = 0$.

Define the form of disturbances in the action of the operators F and G:

$$(Fu)(t) = (F_0 u)(t) + \Delta F \cdot u(t), \ \ t \in [0, T],$$

$$(Gu)(t_j) = (G_0 u)(t_j) + \Delta G_j \cdot \int_0^{t_j} u(s)\, ds, \ \ j = 1, \dots, \mu.$$

Here ΔF and ΔG_j are matrices of dimension $n \times r$ and $\nu \times r$, respectively, with the elements ΔF^{ik} and ΔG_j^{ik} being random values distributed uniformly on the segments $[a^{ik}, b^{ik}]$ and $[a_j^{ik}, b_j^{ik}]$, respectively (we write for short $\Delta F^{ik} \sim U^{ik}$ and $\Delta G_j^{ik} \sim U_j^{ik}$). The operators F_0 and G_0 are assumed to be acting with no disturbances.

In [9], a component-by-component probabilistic description is obtained for the components of $x(t)$ and $z(t_j)$. This description is given in the form of a set of probability density functions parametrized by the current time. To construct these functions, the previously obtained representation of the Cauchy operator of the system under consideration is used.

The system (2.1), (2.2) is a particular case of the general continuous-discrete system considered in [5]. Theorem 1 [5] gives the presentation to solution of (2.1), (2.2) with zero initial values:

$$\begin{pmatrix} x \\ z \end{pmatrix} = \mathcal{C} \begin{pmatrix} Fu \\ Gu \end{pmatrix} = \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} \\ \mathcal{C}_{21} & \mathcal{C}_{22} \end{pmatrix} \begin{pmatrix} Fu \\ Gu \end{pmatrix}, \tag{2.5}$$

where $z = \operatorname{col}(z(t_1), \dots, z(t_\mu))$, $\mathcal{C}$ is the Cauchy operator with block components $\mathcal{C}_{ij}$, $i, j = 1, 2$.

As applied to the case under consideration, the explicit representation of $\mathcal{C}_{ij}$ in the terms of matrix parameters of (2.1), (2.2) is obtained in [6]. In the sequel, we use the components

$$(\mathcal{C}_{11} f)(t) = \int_0^t C_{11}(t, s) f(s)\, ds, \quad (\mathcal{C}_{12} g)(t) = \int_0^t \sum_{j:\, t_j < t} C_{12}(t_j, s) g(t_j)\, ds, \ \ t \in [0, T],$$

$$\mathcal{C}_{21}^i f = \int_0^{t_i} \sum_{j<i} C_{21}^i(t_j, s) f(s)\, ds, \quad \mathcal{C}_{22}^i g = \sum_{j \le i} C_{22}^i(j) g(t_j), \ \ i = 1, \dots, \mu.$$

Here the upper index in notations $\mathcal{C}_{22}^i$ and $\mathcal{C}_{22}^i$ stands for the number of a ν-column in a column from $\mathbb{R}^{\nu\mu}$.

Each component of the solution (x,z) includes the determined term $x_i^0(t)$, $z_i^0(t_j)$ correspondingly to operators F_0 and G_0 and the random term $\xi_i(t)$, $\eta_i(t_j)$ that corresponds to matrices ΔF and ΔG_j:

$$x_i(t)=x_i^0(t)+\xi_i(t),\quad z_i(t_j)=z_i^0(t_j)+\eta_i(t_j).$$

Thus for $\ell x=\operatorname{col}(\ell_1(x,z),\dots,\ell_N(x,z))$ we have

$$\ell_i(x,z)=\ell_i(x^0,z^0)+\ell_i(\xi,\eta),$$

and we are aimed at the description of the distribution to random values $\lambda_i\equiv\ell_i(\xi,\eta)$.

Let us recall the general form of $\ell: AC^n\times FD^\nu(\mu)\to\mathbb{R}^N$:

$$\ell(x,z)=\Psi x(0)+\int\limits_0^T\Phi(s)\dot{x}(s)\,ds+\sum_{j=0}^{\mu}\Gamma_j z(t_j),$$

covering various special cases of target vector-functionals such as multipoint, integral and many others.

Due to (2.5), we have

$$\xi_i(t)=\sum_{\ell=1}^{n}\sum_{k=1}^{r}{}^{11}\theta_{\ell k}^i(t)\,\Delta F^{\ell k}+\sum_{j:\,t_j<t}\sum_{\ell=1}^{\nu}\sum_{k=1}^{r}{}^{12}\theta_{j\ell k}^i(t)\,\Delta G_j^{\ell k} \tag{2.6}$$

and

$$\eta_i(t_j)=\sum_{m=1}^{n}\sum_{k=1}^{r}{}^{21}\theta_{mk}^{ij}\Delta F^{mk}+\sum_{m=1}^{\nu}\sum_{k=1}^{r}\sum_{\ell=1}^{\mu}{}^{22}\theta_{\ell mk}^{ij}\Delta G_\ell^{mk}, \tag{2.7}$$

where matrices ${}^{11}\theta^{ij}$, ${}^{12}\theta^{ij}$, ${}^{21}\theta^{ij}$, ${}^{22}\theta^{ij}$ are defined in [9].

In the way described in [9] we rewrite (2.6), (2.7) in the form

$$\xi_i(t)=\sum_{q=1}^{N}\varphi_q^i(t)\cdot(b_q-a_q)\cdot c_q+\sigma_i(t),\quad \sigma_i(t)=\sum_{q=1}^{N}\varphi_q^i(t)\cdot a_q$$

and

$$\eta_i(t_j)=\sum_{q=1}^{N}\psi_q^i(t_j)\cdot(b_q-a_q)\cdot c_q+\omega_i(t_j),\quad \omega_i(t_j)=\sum_{q=1}^{N}\psi_q^i(t_j)\cdot a_q,$$

where $N=n\cdot r+\nu\cdot\mu\cdot r$.

In the sequel, we will use the component-wise representation of the target vector-functional ℓ:

$$\ell_i(x,z)={}^1\ell_i(x)+{}^2\ell_i(z)=\sum_{j=1}^{n}{}^1\ell_i^j(x_j)+\sum_{j=1}^{\nu}\sum_{k=0}^{\mu}{}^2\ell_i^j(z_j(t_k)).$$

Hence it follows that

$$\lambda_i=\ell_i(\xi,\eta)=\sum_{q=1}^{N}\varkappa_i^q\cdot(b_q-a_q)\cdot c_q+\gamma_i, \tag{2.8}$$

where

$$\varkappa_i^q=\sum_{j=1}^{n}{}^1\ell_i^j(\varphi_q^j)+\sum_{j=1}^{\nu}\sum_{k=0}^{\mu}{}^2\ell_i^{jk}(\psi_q^j(t_k)),\quad \gamma_i=\sum_{j=1}^{n}{}^1\ell_i^j(\sigma_j)+\sum_{j=1}^{\nu}\sum_{k=0}^{\mu}{}^2\ell_i^{jk}(\omega_j^i(t_k)). \tag{2.9}$$

3 Main result

For any $y_1 \in \mathbb{R}$, we define in $\mathbb{R}^{N-1}$ the polyhedral set $\mathcal{M}_i(y_1)$:

$$\mathcal{M}_i(y_1) = \Big\{(y_2, \ldots, y_N)' \in \mathbb{R}^{N-1}: \ \ 0 \le y_q \le 1, \ \ q = 2, \ldots, N;$$
$$\frac{1}{\varkappa_i^1 \cdot (b_1 - a_1)} \cdot y_1 - 1 \le \sum_{q=2}^{N} \frac{\varkappa_i^q \cdot (b_q - a_q)}{\varkappa_i^1 \cdot (b_1 - a_1)} \cdot y_q \le \frac{1}{\varkappa_i^1 \cdot (b_1 - a_1)} \cdot y_1 \Big\}.$$

Theorem. *Let $\varkappa_i^q$, $i = 1, \ldots, n$, $q = 1, \ldots, N$, and γ_i, $i = 1, \ldots, n$, be defined by equalities* (2.9)*, and $\varkappa_i^1 \neq 0$. Then the probability density function $f_{\lambda_i}(y_1)$ of the random variable* (2.8) *is defined by the equality*

$$f_{\lambda_i}(y_1) = \frac{\mathbf{V}^{N-1}[\mathcal{M}_i(y_1 - \gamma_1)]}{|\varkappa_i^1| \cdot (b_1 - a_1)},$$

where $\mathbf{V}^{N-1}[\mathcal{M}]$ is the Lebesgue measure of a set $\mathcal{M} \subset \mathbb{R}^{N-1}$.

Emphasize in conclusion that this result allows to find a segment I_i of all possible values for each component of the target vector-functional and calculate the probability $P(\lambda_i \in J_i)$ for any subset $J_i \subset I_i$. This can be useful when studying control problems with a given target set (see, for instance, [3] and the references therein).

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