Some Tests for Regularity and Almost Reducibility for Limit Periodic Linear Systems

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Consider a linear differential system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^2, \quad t \in \mathbb{R}, \tag{1}$$

with piecewise continuous and bounded coefficient matrix A of the form

$$A(t) = \sum_{k=0}^{+\infty} A_k(t), \tag{2}$$

where A_k, $k = 0, \dots, +\infty$, are periodic matrices with the periods T_k. If each matrix A_k is everywhere continuous and series (2) converges uniformly on the entire time axis $\mathbb{R}$, then the matrix A is limit-periodic [1, p. 32] and, therefore, almost periodic. The problem on Lyapunov regularity of linear systems with almost periodic coefficients was posed by N. P. Erugin at a mathematical seminar at the Institute of Physics and Mathematics of Byelorussian Academy of Sciences in 1956. The formulation of this problem was published in [3, pp. 121, 137], see also [4].

In [6], using some results of [5] V. M. Millionshchikov has proved the existence of some Lyapunov-irregular linear system with limit periodic coefficients. To this end V. M. Millionshchikov has introduced some special class of linear systems. A comprehensive study of systems from Millionshchikov class was made by A. V. Lipntskii in [7–14]. In particular, an explicit example of Lyapunov-irregular system from the Millionshchikov class is given in [7], see also [17].

On the other hand, it is well known [5,15,16], that the set of Lyapunov-regular (and even almost reducible, for the definition of almost reducibility see [2]) systems with almost periodic coefficients is large in some natural sense. However no effective tools to recognize these properties are known.

Our aim here is to give some sufficient conditions for linear systems from Millionshchikov class to be Lyapunov regular or almost reducible. The conditions of regularity and almost reducibility provided by Theorem 1 below are not coefficient, but may be useful in constructing systems from Millionshchikov class with prescribed asymptotic properties.

In what follows we suppose that $T_0 = 2$, $T_k \in \mathbb{N}$, and $T_{k+1}/T_k = m_i \in \mathbb{N}$ for all $k = 0, \dots, +\infty$. We also suppose that $m_k > 1$, $k = 0, \dots, +\infty$. Let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Take some continuous function $\omega : [0,1] \to \mathbb{R}$ such that $\omega(0) = \omega(1) = 0$ and $\int\limits_0^1 \omega(t)\,dt = 1$. Take also a sequence $\varphi : \mathbb{N} \to [0, \pi/2[$. As usually, the values of the sequence φ we denote by φ_k, $k \in \mathbb{N}$.

Now let us define the matrices A_k by the following equalities:

$$A_0(t) = \begin{cases} \omega(t)D, & \text{for } t \in [0,1[\,, \\ 0, & \text{for } t \in [1,2[\,, \end{cases} \tag{3}$$

for $k=0$ and

$$A_k(t)=\begin{cases}-\varphi_k\omega(t)J, & \text{for } t\in[0,1[\,,\\ 0, & \text{for } t\in[1,T_i[\,,\end{cases} \tag{4}$$

for all $k=1,\dots,+\infty$.

Lemma 1. *If* $\sum\limits_{k=1}^{\infty}\varphi_k<+\infty$, *then system* (1) *with the coefficient matrix A defined by* (3) *and* (4) *is limit periodic.*

Let $S_m(t)=\sum\limits_{k=0}^{m}A_k(t)$, $m=1,\dots,+\infty$, where A_k are defined by (3) and (4). It can be easily seen that each matrix S_m is T_m-periodic. Now for arbitrary $m\in\mathbb{N}$ consider a periodic linear system

$$\dot z=S_m(t)z,\quad z\in\mathbb{R}^2,\quad t\in\mathbb{R}. \tag{5}$$

Denote the Cauchy matrix of system (5) by Z_m. Then the monodromy matrix of system (5) can be written as $Z_m(T_m,0)$. Hence the eigenvalues of $Z_m(T_m,0)$ are the Floquet multipliers of system (5).

Definition. We say that system (1) with the coefficient matrix A defined by (3) and (4) is a real-type system if all Floquet multipliers of each corresponding system (5) with $m\in\mathbb{N}$ are real.

Remark. Note that the condition $\varphi_k\in[0,\pi/2[$ guarantees that the Floquet multipliers of system (5) are positive.

Lemma 2. *If system* (1) *with the coefficient matrix A defined by* (3) *and* (4) *is a real-type system, then all eigenvectors of matrices $Z_m(T_m,0)$, $m\in\mathbb{N}$ lie in the first quadrant, i.e. have positive coordinates.*

Suppose that system (1) with the coefficient matrix A defined by (3) and (4) is a real-type system. Let ζ_1^m and ζ_2^m be some eigenvectors of $Z_m(T_m,0)$, where each vector ζ_2^m corresponds to greater eigenvalue of $Z_m(T_m,0)$. Denote the angle between ζ_1^m and ζ_2^m by β_m.

Theorem 1. *The following statements are valid:*

(i) *If the angle β_k is separated from zero, then system* (1) *is almost reducible.*

(ii) *If* $\varliminf\limits_{k\to\infty}T_k^{-1}\ln\beta_k=0$, *then system* (1) *is Lyapunov regular.*

To prove the first statement we use the fact that system (1) lies in the closure of the set of reducible systems. The second statement is based on the following lemma.

Lemma 3. *Let x_{mj} be the solution of system* (1) *satisfying the condition $x_{mj}(jT_m)=\zeta_2^m$ for some $j\in\mathbb{N}$. Then the vectors $x_{mj}(t)$ lie between ζ_1^m and ζ_2^m for all $t=(j+l)T_m$, $l\in\mathbb{N}$.*

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