Branches of Positive Solutions of a Superlinear Indefinite Prescribed Curvature Problem

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1 The result

This contribution is based on our recent paper [4] where we analyzed the set of positive regular solutions of the quasilinear Neumann problem

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1+(u')^2}}\Big)' = \lambda a(x) f(u), \quad 0 < x < 1, \\ u'(0) = u'(1) = 0. \end{cases} \tag{1.1}$$

Here, $\lambda \in \mathbb{R}$ is a parameter and the functions a and f satisfy:

(a_1) $a \in L^\infty(0,1)$, $\int\limits_0^1 a(x)\,dx < 0$, *and there is* $z \in (0,1)$ *such that* $a(x) > 0$ *almost everywhere in* $(0,z)$ *and* $a(x) < 0$ *almost everywhere in* $(z,1)$*;*

(f_1) $f \in \mathcal{C}^0[0,+\infty)$, $f(s) > 0$ *if* $s > 0$, *and, for some constant* $p > 1$, $\lim\limits_{s\to 0^+} \frac{f(s)}{s^p} = 1$.

As a is sign indefinite and f is superlinear at zero, (1.1) is a *superlinear indefinite* elliptic problem. These problems have attracted a huge amount of attention during the last few decades.

The problem (1.1) can be regarded as a simple prototype of its more sophisticated multidimensional counterpart, which plays a central role in the mathematical analysis of a number of important geometrical and physical issues, ranging from prescribed mean curvature problems for cartesian surfaces in the Euclidean space, to the study of capillarity phenomena for compressible or incompressible fluids, as well as to the analysis of reaction-diffusion processes where the flux features saturation at high regimes.

Although the study of (1.1) is often settled in the space of *bounded variation* functions (see, e.g., [5–8]), here we will be instead concerned with the *regular* solutions of (1.1), that is, functions $u \in W^{2,1}(0,1)$ which fulfill the differential equation almost everywhere in $(0,1)$, as well as the boundary conditions.

A function $u \in \mathcal{C}^0[0,1]$ is said to be *positive* if $\min\limits_{[0,1]} u \geq 0$ and $\max\limits_{[0,1]} u > 0$, whereas it is said *strictly positive* if $\min\limits_{[0,1]} u > 0$. Here, the positive solutions of (1.1) are regarded as couples (λ, u). Naturally, for any given $\lambda \geq 0$, a couple (λ, u) is said to be a positive, or strictly positive, solution of (1.1) if u is a positive, or strictly positive, solution of (1.1), respectively. Note that, under conditions (a_1) and (f_1), the strong maximum principle (see, e.g., [5, Theorem 2.1]) yields the strict positivity of any positive regular solution of (1.1).

Subsequently, we denote by $\mathscr{S}^+$ the set of all couples $(\lambda, u) \in [0, \infty) \times C^1[0,1]$ such that (λ, u) is a positive, and hence strictly positive, regular solution of (1.1).

The following result establishes the existence of an unbounded closed connected subset $\mathscr{C}^+$ of $\mathscr{S}^+$, bifurcating from $u = 0$ as $\lambda \to +\infty$, and provides simultaneously some sharp information on its localization. The existence of unstable solutions, however not necessarily belonging to $\mathscr{C}^+$, is also detected.

Theorem 1.1. *Assume (a_1) and (f_1). Then, there exists an unbounded closed connected subset $\mathscr{C}^+$ of $\mathscr{S}^+$ for which the following properties hold:*

(i) *there is $\lambda^* > 0$ such that $[\lambda^*, \infty) \subseteq \operatorname{proj}_{\mathbb{R}}(\mathscr{C}^+)$;*

(ii) *there are functions α and β, explicitly defined by* (2.6) *and* (2.7) *respectively, such that, for every $(\lambda, u_\lambda) \in \mathscr{C}^+$, one has*

$$u_\lambda(x_\lambda) < \lambda^{\frac{1}{1-p}} \alpha(x_\lambda), \ \ \textit{for some} \ \ x_\lambda \in [0, z),$$

and

$$u_\lambda(y_\lambda) > \lambda^{\frac{1}{1-p}} \beta(y_\lambda), \ \ \textit{for some} \ \ y_\lambda \in [0, 1];$$

(iii) *there is $C > 0$ such that, for every $(\lambda, u\lambda) \in \mathscr{C}^+$,*

$$\|u_\lambda'\|_{L^\infty(0,1)} < C\lambda^{\frac{1}{1-p}}.$$

Moreover, for every $\lambda \in [\lambda_*, \infty)$, there exists at least one Lyapunov unstable solution $u \in \mathscr{S}^+$ of (1.1) *satisfying the conditions expressed by properties* (ii) *and* (iii)*.*

Theorem 1.1 is a substantial sharpening of some previous results obtained in [6–8]. Unlike in these papers, here the proof exploits an alternative method based on viewing (1.1) as a perturbation of a semilinear problem, on constructing some non-ordered lower and upper solutions, and on using the Leray–Schauder degree. This approach, which appears of interest in its own, yields, in addition, the localization and the instability information established by Theorem 1.1, which is a novel result in the context of the problem (1.1).

2 The proof

2.1 Reformulation of (1.1) as a perturbation of a semilinear problem

Since $f(0) = 0$ and we are focusing attention on the positive solutions of (1.1), without loss of generality we can extend f to the whole of $\mathbb{R}$ as an even function. By performing the change of variable

$$u = \varepsilon v, \ \ \varepsilon = \lambda^{\frac{1}{1-p}}, \tag{2.1}$$

and setting

$$h(s) = \begin{cases} \dfrac{f(s)}{|s|^p} & \text{if } \ s \neq 0, \\ 1 & \text{if } \ s = 0, \end{cases} \tag{2.2}$$

the problem (1.1) can be equivalently written in the form

$$\begin{cases} -v'' = a(x)|v|^p\, h(\varepsilon v)\,(1+(\varepsilon v')^2)^{\frac{3}{2}}, \quad 0<x<1,\\ v'(0)=v'(1)=0. \end{cases} \tag{2.3}$$

Throughout the rest of this proof, for every $r>0$, we consider the auxiliary function

$$\ell_r(x,s) = \begin{cases} |s|^p & \text{if } s\le 0,\\ a(x)\,s^p & \text{if } 0<s\le r,\\ a(x)\,s^p\,(r+1-s) & \text{if } r<s\le r+1,\\ -s+r+1 & \text{if } s>r+1, \end{cases}$$

as well as the associated problem

$$\begin{cases} -v'' = \ell_r(x,v)\, h(\varepsilon v)\,(1+(\varepsilon v')^2)^{\frac{3}{2}}, \quad 0<x<1,\\ v'(0)=v'(1)=0. \end{cases} \tag{2.4}$$

It is obvious that any solution v of (2.4), with $0\le v\le r$ in $[0,1]$, solves (2.3). Moreover, due to (2.2), the problem (2.4) perturbs, as $\varepsilon>0$ separates away from 0, from the semilinear x problem

$$\begin{cases} -v'' = \ell_r(x,v), \qquad 0<x<1,\\ v'(0)=v'(1)=0. \end{cases} \tag{2.5}$$

2.2 Existence of non-ordered strict lower and upper solutions of (2.5)

Construction of a lower solution α

Let $\mu_1>0$ be the principal eigenvalue of the linear weighted eigenvalue problem

$$\begin{cases} ll-\varphi'' = \mu\, a(x)\,\varphi, \qquad 0<x<\dfrac{z}{2},\\ \varphi'(0)=0,\ \ \varphi\Big(\dfrac{z}{2}\Big)=0. \end{cases}$$

Denote by φ_1 any positive eigenfunction associated to μ_1 and let $\overline{x}\in(0,\frac{z}{2})$ be such that

$$\varphi_1(\overline{x})+\varphi_1'(\overline{x})(x-\overline{x})=0.$$

Then, we define, for $c>0$,

$$\alpha(x) = \begin{cases} c\varphi_1(x) & \text{if } 0\le x<\overline{x},\\ c\varphi_1(\overline{x})+c\varphi_1'(\overline{x})(x-\overline{x}) & \text{if } \overline{x}\le x<z,\\ 0 & \text{if } z\le x\le 1. \end{cases} \tag{2.6}$$

Construction of an upper solution β

For every $\kappa>0$, let us denote by z_κ the unique solution of the linear problem

$$\begin{cases} -z'' = \Big(a(x)-\displaystyle\int_0^1 a(t)\,dt\Big)\kappa^p, \qquad 0<x<1,\\ z'(0)=z'(1)=0,\ \ \displaystyle\int_0^1 z(t)\,dt=0. \end{cases}$$

Then, we define

$$\beta = z_\kappa + \kappa. \tag{2.7}$$

By making a suitable choice of c and the following conclusions about α and β can be inferred.

Proposition 2.1. *There exists a constant $r_0 > 0$ such that, for all $r \geq r_0$, the problem* (2.5) *admits a lower solution α and an upper solution β, respectively defined by* (2.6) *and* (2.7)*, such that:*

(i) *$\beta - \alpha$ changes sign in $[0, 1]$;*

(ii) *any solution v of* (2.5) *such that $\alpha \leq v$ in $[0, 1]$, satisfies $\alpha(x) < v(x)$ for all $x \in [0, 1]$;*

(iii) *any solution v of* (2.5) *such that $v \leq \beta$ in $[0, 1]$, satisfies $v(x) < \beta(x)$ for all $x \in [0, 1]$.*

2.3 Positivity and a priori bounds for the solutions of (2.5)

Proposition 2.2. *Fix any $r > 0$. Then, the following assertions hold:*

(i) *every solution of* (2.5) *is non-negative;*

(ii) *every positive solution of* (2.5) *is strictly positive.*

Proposition 2.3. *The following assertions hold:*

(i) *for every $r > 0$, any solution v of* (2.5) *satisfies*

$$0 \leq v(x) \leq r + 1, \;\; \textit{for all} \;\; x \in [0, 1],$$

and

$$\|v'\|_{L^\infty(0,1)} < C = \|a\|_{L^1(0,1)}(r + 1)^{p+1}; \tag{2.8}$$

(ii) *for every $r \geq r_0$, any solution v of* (2.5)*, with $v(x_0) \leq \alpha(x_0)$ for some $x_0 \in [0, 1]$, satisfies*

$$\max_{[0,1]} v < R = \|\alpha\|_{L^\infty(0,1)} + \|\alpha'\|_{L^\infty(0,1)}. \tag{2.9}$$

2.4 Existence of ordered strict lower and upper solutions of (2.5)

Proposition 2.4. *Fix any $r \geq r_0$. The constants $\alpha_1 = -1$ and $\beta_1 = r + 2$ are, respectively, a lower solution and an upper solution of* (2.5) *satisfying*

$$\alpha_1 < 0 \leq \alpha(x), \;\; \beta(x) \leq r_0 < \beta_1, \;\; \textit{for all} \;\; x \in [0, 1]. \tag{2.10}$$

Moreover, every solution v of (2.5) *is such that $\alpha_1 < v(x) < \beta_1$, for all $x \in [0, 1]$.*

2.5 Degree computations

Fix any $r \geq \max\{r_0, R\}$, where R is the constant defined in (2.9). Then, C being the constant introduced in (2.8), define the following open bounded subsets of $C^1[0, 1]$:

$$\begin{aligned}
\Omega_1 &= \Big\{ v \in \mathcal{C}^1[0,1] : \;\; \alpha_1 < v(x) < \beta_1 \;\text{ for all }\; x \in [0,1], \;\; \|v'\|_\infty < C \Big\}, \\
\Omega_2 &= \Big\{ v \in \mathcal{C}^1[0,1] : \;\; \alpha_1 < v(x) < \beta(x) \;\text{ for all }\; x \in [0,1], \;\; \|v'\|_\infty < C \Big\}, \\
\Omega_3 &= \Big\{ v \in \mathcal{C}^1[0,1] : \;\; \alpha(x) < v(x) < \beta_1 \;\text{ for all }\; x \in [0,1], \;\; \|v'\|_\infty < C \Big\}, \\
\Omega = \Omega_1 \setminus \overline{\Omega_2 \cup \Omega_3} &= \Big\{ v \in \Omega_1 : \;\; v(x_0) < \alpha(x_0) \;\text{ and }\; \beta(y_0) < v(y_0) \;\text{ for some }\; x_0, y_0 \in [0,1] \Big\}.
\end{aligned}$$

From (2.10), it follows that $\Omega_2 \cup \Omega_3 \subset \Omega_1$. Moreover, we have that $\Omega_2 \cap \Omega_3 = \varnothing$ by Proposition 2.1.

Let us denote by $\mathcal{T} : [0,\infty)\times\mathcal{C}^1[0,1] \to \mathcal{C}^1[0,1]$ the operator sending each $(\varepsilon, v) \in [0,\infty)\times\mathcal{C}^1[0,1]$ to the unique solution $w \in W^{2,\infty}(0,1)$ of the linear problem

$$\begin{cases} -w'' + w = \ell_r(x,v)\, h(\varepsilon v)\,(1+(\varepsilon v')^2)^{\frac{3}{2}} + v, \quad 0 < x < 1, \\ w'(0) = w'(1) = 0. \end{cases}$$

It is clear that $\mathcal{T}$ is completely continuous and that its fixed points are the solutions of the problem (2.4). Moreover, by Propositions 2.1 and 2.3 and our choice of C, the operator $T(0,\,\cdot\,)$ cannot have fixed points on $\partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_3$. Thus, by the additivity property of the degree, we infer that

$$\begin{aligned} &\deg_{LS}\big(\mathcal{I} - \mathcal{T}(0,\,\cdot\,), \mathcal{O}\big) \\ &\qquad = \deg_{LS}\big(\mathcal{I} - \mathcal{T}(0,\,\cdot\,), \Omega_1\big) - \deg_{LS}\big(\mathcal{I} - \mathcal{T}(0,\,\cdot\,), \Omega_2\big) - \deg_{LS}\big(\mathcal{I} - \mathcal{T}(0,\,\cdot\,), \Omega_3\big). \end{aligned}$$

As, from, e.g., [1, Chapter III], we know that $\deg_{LS}(\mathcal{I} - \mathcal{T}(0,\,\cdot\,), \Omega_i) = 1$, for $i = 1, 2, 3$, we can conclude that $\deg_{LS}(\mathcal{I} - \mathcal{T}(0,\,\cdot\,), \Omega) = -1$. Therefore, by the existence property of the degree, the problem (2.5) possesses a solution $v \in \Omega$, where necessarily $x_0 \in [0, z)$, because $\alpha(x_0) > v(x_0) > 0$ and $\alpha = 0$ on $[z, 1]$. In addition, having chosen $r > R$, Proposition 2.3 guarantees that $v(x) < r$ for all $x \in [0,1]$ and therefore v is a solution of the problem (2.3) for $\varepsilon = 0$. Hence, if we define

$$\mathcal{O} = \Big\{ v \in \Omega : \ \min_{[0,1]} v > 0, \ \max_{[0,1]} v < r \Big\},$$

then every solution $v \in \Omega$ must belong to $\mathcal{O}$. Thus, the excision property of the degree yields

$$\deg_{LS}\big(\mathcal{I} - \mathcal{T}(0,\,\cdot\,), \mathcal{O}\big) = -1.$$

2.6 Existence of a continuum and conclusion of the proof

The boundedness of $\partial\mathcal{O}$ in $\mathcal{C}^1[0,1]$ and the complete continuity of the operator $\mathcal{T}$ guarantee the existence of some $\varepsilon^* > 0$ such that $\mathcal{T}(\varepsilon,\,\cdot\,)$ has no fixed points on $\partial\mathcal{O}$ for all $\varepsilon \in [0, \varepsilon^*]$. Consequently, the homotopy property of the degree implies that $\deg_{LS}(\mathcal{I} - \mathcal{T}(0,\varepsilon), \mathcal{O}) = -1$ for all $\varepsilon \in [0,\varepsilon^*]$, and hence the existence of at least one solution $v = v_\varepsilon \in \mathcal{O}$ of the problem (2.3) for all $\varepsilon \in [0, \varepsilon^*]$. Actually, the Leray–Schauder continuation theorem [3, p. 63] provides us with a continuum $\mathscr{K}^+$ of solutions $(\varepsilon, v_\varepsilon)$ of (2.3) with $\varepsilon \in [0, \varepsilon^*]$ and $v_\varepsilon \in \mathcal{O}$.

The change of variables (2.1) then implies the existence of a closed connected set $\mathscr{C}^+$ of solutions (λ, u_λ) of (1.1), where $\lambda = \varepsilon^{1-p} \in [\lambda_*, \infty)$, with $\lambda_* = (\varepsilon^*)^{1-p}$, and

$$u_\lambda = \varepsilon v_\varepsilon = \lambda^{\frac{1}{1-p}} v_\varepsilon.$$

It is apparent that every $(\lambda, u_\lambda) \in \mathscr{C}^+$ is strictly positive and satisfies conditions (ii) and (iii).

Finally, adapting the results in [2], we can prove the existence, for each $\varepsilon \in [0, \varepsilon^*]$, of a Lyapunov unstable solution $v \in \mathcal{O}$ of (2.4). Consequently, for every $\lambda \in [\lambda_*, \infty)$ there is at least one unstable solution u_λ of (1.1) which is strictly positive and satisfies (ii) and (iii). This concludes the proof of Theorem 1.1.

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