# Periodic Solutions of a Pendulum Like Planar System

**Alexander Lomtatidze**

*Institute of Mathematics, Faculty of Mechanical Engineering, Brno University of Technology Brno, Czech Republic*

*E-mail:* lomtatidze@fme.vutbr.cz

We consider the periodic problem

$$u' = f(t, u, v), \;\; v' = p(t) \sin u + q(t); \quad u(0) = u(\omega), \;\; v(0) = v(\omega). \tag{1}$$

Here we assume that $p, q \in L([0, \omega])$, $p \not\equiv 0$, and $f \in Car([0, \omega] \times \mathbb{R}^2)$ satisfies the conditions

$$f(t, x, y) \operatorname{sgn} y \geq 0, \quad |f(t, x, y)| \leq h(t, |y|) \quad \text{for } \; t \in [0, \omega], \;\; x, y \in \mathbb{R},$$

where $h \in Car([0, \omega] \times \mathbb{R}_+)$ is non-decreasing in the second argument.

The theory of BVPs for non-autonomous and non-resonant systems is quite well developed (see, [3]). However, (1) is a resonant type problem. Some particular cases of (1) are studied in the literature, but usually under the assumption that $\int\limits_0^\omega q(s)\,\mathrm{d}s = 0$ (see, e.g., [2, 4]). As for the case $\int\limits_0^\omega q(s)\,\mathrm{d}s \neq 0$, there are only a few results available in the existing literature (see, [1, 5]). Below we present new results concerning the existence, multiplicity, and localization of solutions of (1).

We use the following notation:

$$[x]_\pm = \frac{1}{2}\,(|x| \pm x),$$

$$q_0(t) = \max\big\{\|[q]_+\|_L, \|[q]_-\|_L\big\}, \quad H(y) = \int\limits_0^\omega h(s, |y|)\,\mathrm{d}s,$$

$$a(\ell) = \frac{\pi}{2} + \frac{1}{4}\,H(\ell), \quad b(\ell) = \frac{\pi}{2} - \frac{1}{4}\,H(\ell),$$

$$I_{ak}(\ell) = \,]-a(\ell) + 2k\pi, a(\ell) + 2k\pi[\,, \;\; I_{bk}(\ell) = \,]-b(\ell) + 2k\pi, b(\ell) + 2k\pi[\,,$$

$$J_{ak}(\ell) = \,]-a(\ell) + (2k+1)\pi, a(\ell) + (2k+1)\pi[\,, \;\; I_{bk}(\ell) = \,]-b(\ell) + (2k+1)\pi, b(\ell) + (2k+1)\pi[\,,$$

$$B(\ell) = \Big\{v \in C([0, \omega]) : \;\; \|v\|_C \leq \ell, \;\; v(t_0) = 0 \; \text{ for some } \; t_0 \in [0, \omega]\Big\}.$$

**Theorem 1.** *Let $\sigma \in \{-1, 1\}$, $\ell \stackrel{\text{def}}{=} \|[\sigma p]_-\|_L + q_0$, and the conditions*

$$H(\ell) < 2\pi, \tag{2}$$

$$\|[\sigma p]_-\|_L + \bigg|\int\limits_0^\omega q(s)\,\mathrm{d}s\bigg| \leq \|[\sigma p]_+\|_L \cos\frac{H(\ell)}{4} \tag{3}$$

*hold. Then, for any $k \in \mathbb{Z}$, problem (1) possesses a solution $(u_k, v_k)$ such that $v_k \in B(\ell)$, and*

$$\operatorname{Range} u_k \subseteq \overline{I_{ak}(\ell)}, \quad \overline{I_{bk}(\ell)} \cap \operatorname{Range} u_k \neq \varnothing \quad \textit{if } \; \sigma = 1,$$

*and*

$$\operatorname{Range} u_k \subseteq \overline{J_{ak}(\ell)}, \quad \overline{J_{bk}(\ell)} \cap \operatorname{Range} u_k \neq \varnothing \quad \textit{if } \; \sigma = -1.$$

**Remark 1.** Inequality (3) is optimal for the solvablity of (1) and cannot be replaced by

$$\|[\sigma p]_-\|_L + \left|\int_0^\omega q(s)\,\mathrm{d}s\right| \leq (1+\varepsilon)\|[\sigma p]_+\|_L \cos\frac{H(\ell)}{4}\,,$$

no matter how small $\varepsilon > 0$ is.

Theorem 1 guarantees that problem (1) possesses infinitely many solutions $(u_k, v_k)$. However, it may happen that $u_{k+1} \equiv u_k + 2\pi$ (for example, if $f(t, x+2\pi, y) \equiv f(t, x, y)$). Introduce the following definition.

**Definition 1.** Solutions $(u_1, v_1)$ and $(u_2, v_2)$ of (1) are said to be geometrically distinct (g.d.) if $u_1 - u_2 \not\equiv 2\pi n$ for $n \in \mathbb{Z}$.

**Theorem 2.** *Let $\sigma \in \{-1, 1\}$, $\ell \overset{\text{def}}{=} \frac{1}{2}(\|p]\|_L + \|q]\|_L)$, inequality* (2) *hold, and*

$$\|[\sigma p]_-\|_L + \left|\int_0^\omega q(s)\,\mathrm{d}s\right| < \|[\sigma p]_+\|_L \cos\frac{H(\ell)}{4}\,.$$

*Then, for any $k \in \mathbb{Z}$, problem* (1) *possesses a pair of g.d. solutions $(u_{1k}, v_{1k})$ and $(u_{2k}, v_{2k})$ such that $v_{ik} \in B(\ell)$ for $i = 1, 2$, and*

$$\operatorname{Range} u_{1k} \subset I_{ak}(\ell), \ \ I_{bk}(\ell) \cap \operatorname{Range} u_{1k} \neq \varnothing, \ \ \textit{and} \ \ \operatorname{Range} u_{2k} \subset J_{ak}(\ell), \ \ J_{bk}(\ell) \cap \operatorname{Range} u_{2k} \neq \varnothing.$$

**Definition 2.** Solutions $(u_1, v_1)$ and $(u_2, v_2)$ of (1) are said to be consecutive if $u_1(t) \leq u_2(t)$ for $t \in [0, \omega]$, $u_1 \not\equiv u_2$, and problem (1) has no solution $(u, v)$ satisfying $u_1(t) \leq u(t) \leq u_2(t)$ for $t \in [0, \omega]$, $u \not\equiv u_1$, and $u \not\equiv u_2$.

It is worth mentioning that a pair of consecutive solutions may not be geometrically distinct and vice versa.

In order to formulate the next theorem, we need to introduce the following hypothesis:

$$\left.\begin{array}{c} \text{the function } \ f(t,x,\cdot) : \mathbb{R} \to \mathbb{R} \ \text{ is non-decreasing for a.e. } \ t \in [0,\omega] \ \text{ and all } \ x \in \mathbb{R}, \\ \operatorname{mes}\big\{t \in [0,\omega] : \ f(t,x,y) \neq 0\big\} > 0 \ \text{ for } \ x, y \in \mathbb{R}, \ \ y \neq 0, \\ \text{for every } \ \varepsilon > 0 \ \text{ and } \ r > 0 \ \text{ there exists } \ f_{\varepsilon r} \in L([0,\omega]) \ \text{ such that} \\ |f(t,x_2,y) - f(t,x_1,y)| \leq f_{\varepsilon r}(t) \ \text{ for } \ t \in [0,\omega], \ \ |x_2 - x_1| \leq \varepsilon, \ \ |y| \leq r. \end{array}\right\} \quad (\mathrm{A})$$

**Theorem 3.** *Let $\sigma \in \{-1, 1\}$, $\ell \overset{\text{def}}{=} \|[\sigma p]_-\|_L + q_0$, $\ell^* \overset{\text{def}}{=} \frac{1}{2}(\|p]\|_L + \|q]\|_L)$, and hypothesis* (A) *hold. Let, moreover, $H(\ell^*) < \pi$ and*

$$\|[\sigma p]_-\|_L - \|[\sigma p]_+\|_L \cos\frac{H(\ell^*)}{2} < \left|\int_0^\omega q(s)\,\mathrm{d}s\right| \leq \|[\sigma p]_+\|_L \cos\frac{H(\ell)}{4} - \|[\sigma p]_-\|_L.$$

*Then, for any $k \in \mathbb{Z}$, problem* (1) *possesses a pair of consecutive solutions $(u_{1k}, v_{1k})$ and $(u_{2k}, v_{2k})$ such that either $(u_{1k}, v_{1k})$ or $(u_{2k}, v_{2k})$ is Lyapunov unstable, $v_{ik} \in B(\ell)$ for $i = 1, 2$, and*

$$\operatorname{Range}(u_{1k} - 2k\pi) \subseteq \Big[-a(\ell), \frac{\pi}{2}\Big[, \quad \operatorname{Range}(u_{2k} - 2k\pi) \subset \Big]\frac{\pi}{2}, \frac{5\pi}{2}\Big[ \quad \textit{if } \sigma\int_0^\omega q(s)\,\mathrm{d}s \geq 0,$$

*and*

$$\operatorname{Range}(u_{1k}-2k\pi)\subset\Big]-\frac{5\pi}{2}\,,-\frac{\pi}{2}\Big[,\quad \operatorname{Range}(u_{2k}-2k\pi)\subseteq\Big]-\frac{\pi}{2}\,,a(\ell)\Big]\quad \text{if } \sigma\int\limits_0^\omega q(s)\,\mathrm{d}s\leq 0.$$

*Moreover, if, for $i\in\{1,2\}$, the inequality $(-1)^i\sigma\int\limits_0^\omega q(s)\,\mathrm{d}s\geq 0$ holds, then, for every solution $(u,v)$ of problem* (1)*, the condition*

$$\Big\{(-1)^i\,\frac{\pi}{2}+2\pi n:\ n\in\mathbb{Z}\Big\}\cap\operatorname{Range}u\neq\varnothing$$

*is satisfied.*

Now, we consider a particular case of (1), namely, the problem

$$u'=h(t)\varphi(v),\ \ v'=p(t)\sin u+q(t);\quad u(0)=u(\omega),\ \ v(0)=v(\omega), \tag{4}$$

where $\varphi:\mathbb{R}\to\mathbb{R}$ is an increasing continuous function satisfying the conditions $\varphi(-y)=-\varphi(y)$ and $\varphi(y)>0$ for $y>0$, and $h\in L([0,\omega])$ is a non-trivial non-negative function. Moreover, we assume that

$$\varphi^*(x,y)\overset{\text{def}}{=}\frac{\varphi(x)-\varphi(y)}{x-y}\ \text{ is continuous}$$

and we put

$$\varphi^*_r\overset{\text{def}}{=}\max\big\{\varphi^*(x,y):x,y\in[-r,r]\big\}.$$

**Definition 3.** A pair of solutions $(u_1,v_1)$ and $(u_2,v_2)$ of (4) is called a fundamental system of solutions if, for any solution $(u,v)$ of (4), there exists $k\in\mathbb{Z}$ such that either $u\equiv u_1+2k\pi$ or $u\equiv u_2+2k\pi$.

**Theorem 4.** *Let $\sigma\in\{-1,1\}$, $\ell^*\overset{\text{def}}{=}\frac{1}{2}\left(\|p]\|_L+\|q]\|_L\right)$, and*

$$\sigma p(t)\geq 0\quad \text{for } t\in[0,\omega]. \tag{5}$$

*Let, moreover,*

$$\|h\|_L\varphi(\ell^*)<\pi,\quad \varphi^*_{\ell^*}\|h\|_L\|p\|_L\leq 16,\quad \text{and}\quad \bigg|\int\limits_0^\omega q(s)\,\mathrm{d}s\bigg|<\|p\|_L\cos\frac{\|h\|_L\varphi(\ell^*)}{2}\,.$$

*Then, problem* (4) *possesses a fundamental system of solutions $(u_1,v_1)$ and $(u_2,v_2)$ such that $v_1,v_2\in B(\ell^*)$, and*

$$\operatorname{Range}u_1\subset\Big]-\frac{\pi}{2}\,,\frac{\pi}{2}\Big[\quad \text{and}\quad \operatorname{Range}u_2\subset\Big]\frac{\pi}{2}\,,\frac{3\pi}{2}\Big[.$$

*Moreover, for $\sigma=1$, $(u_1,v_1)$ is unstable, while for $\sigma=-1$, $(u_2,v_2)$ is unstable.*

As an example, we consider the so-called relativistic problem

$$u'=h(t)\,\frac{v}{\sqrt{1+v^2}}\,,\ \ v'=p(t)\sin u+q(t);\quad u(0)=u(\omega),\ \ v(0)=v(\omega). \tag{6}$$

It is clear that $H(y)=\|h\|_L\,\frac{|y|}{\sqrt{1+y^2}}$ in this case. Therefore, taking into account that $H(y)<\|h\|_L$ and the monotonicity of the cosine function, we get from Theorem 3 the following corollary.

**Corollary 1.** *Let $\sigma \in \{-1, 1\}$, $\|h\|_L \le 2\pi$, and*

$$\bigg|\int\limits_0^\omega q(s)\,\mathrm{d}s\bigg| \le \|[\sigma p]_+\|_L \cos\frac{\|h\|_L}{4} - \|[\sigma p]_-\|_L .$$

*Then, problem* (6) *possesses a pair of g.d. solutions* $(u_1, v_1)$ *and* $(u_2, v_2)$*. If, moreover,* $\|h\|_L < \pi$ *and*

$$\bigg|\int\limits_0^\omega q(s)\,\mathrm{d}s\bigg| \ge \|[\sigma p]_-\|_L - \|[\sigma p]_+\|_L \cos\frac{\|h\|_L}{4} ,$$

*then* $(u_1, v_1)$ *and* $(u_2, v_2)$ *are consecutive solutions and at least one of them is unstable.*

Theorem 4 implies the following corollary.

**Corollary 2.** *Let $\sigma \in \{-1, 1\}$ and* (5) *be fulfilled. Let, moreover,*

$$\|h\|_L \le \pi, \quad \|h\|_L \|p\|_L \le 16, \quad \text{and} \quad \bigg|\int\limits_0^\omega q(s)\,\mathrm{d}s\bigg| \le \|p\|_L \cos\frac{\|h\|_L}{2} .$$

*Then, the conclusions of Theorem 4 hold for problem* (6).

At last we mention that the above theorems also guarantee a localization of the second component of solutions (see, the conditions like $v \in B(\ell)$). Therefore, our results can be applied to some singular problems as well. For example, let us consider the so-called mean curvature problem

$$u' = f(t, u)\,\frac{v}{\sqrt{1 - v^2}}\,, \quad v' = p(t)\sin u + q(t); \quad u(0) = u(\omega), \ \ v(0) = v(\omega),$$

where $f \in Car([0, \omega] \times \mathbb{R})$ and $0 \le f(t, x) \le h(t)$ for $t \in [0, \omega]$, $x \in \mathbb{R}$. Theorem 1 yields the following corollary.

**Corollary 3.** *Let $\sigma \in \{-1, 1\}$, $\ell \overset{\text{def}}{=} \|[\sigma p]_-\|_L + q_0$, $\ell < 1$, and inequalities* (2) *and* (3) *be satisfied with $H(\ell) \overset{\text{def}}{=} \frac{\|h\|_L \ell}{\sqrt{1-\ell^2}}$ . Then, problem* (6) *possesses infinitely many solutions.*

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## References

[1] C. Bereanu, P. Jebelean and J. Mawhin, Periodic solutions of pendulum-like perturbations of singular and bounded $\phi$-Laplacians. *J. Dynam. Differential Equations* **22** (2010), no. 3, 463–471.

[2] H. Brezis and J. Mawhin, Periodic solutions of the forced relativistic pendulum. *Differential Integral Equations* **23** (2010), no. 9-10, 801–810.

[3] I. T. Kiguradze, Boundary value problems for systems of ordinary differential equations. (Russian) Translated in *J. Soviet Math.* **43** (1988), no. 2, 2259–2339. *Itogi Nauki i Tekhniki, Current problems in mathematics. Newest results, Vol. 30 (Russian)*, 3–103, 204, *Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow*, 1987.

[4] J. Mawhin, Global results for the forced pendulum equation. *Handbook of differential equations*, 533–589, *Elsevier/North-Holland, Amsterdam*, 2004.

[5] G. Tarantello, On the number of solutions for the forced pendulum equation. *J. Differential Equations* **80** (1989), no. 1, 79–93.